RESOLUTIONS FOR METRIZABLE COMPACTA
IN EXTENSION THEORY

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ABSTRACT. We prove a K-resolution theorem for simply connected CW-complexes K in extension theory in the class of metrizable compacta X. This means that if K is a connected CW-complex, G is an abelian group, $n \in \mathbb{N}_{\geq 2}$, $G = \pi_n(K)$, $\pi_k(K) = 0$ for $0 \leq k < n$, and extdim $X \leq K$ (in the sense of extension theory, that is, K is an absolute extensor for X), then there exists a metrizable compactum Z and a surjective map $\pi : Z \to X$ such that:

(a) π is G-acyclic,

(b) $\dim Z \leq n + 1$, and

(c) extdim $Z \leq K$.

This implies the G-resolution theorem for arbitrary abelian groups G for cohomological dimension $\dim_G X \leq n$ when $n \in \mathbb{N}_{\geq 2}$. Thus, in case K is an Eilenberg-Mac Lane complex of type $K(G, n)$, then (c) becomes $\dim_G Z \leq n$.

If in addition $\pi_{n+1}(K) = 0$, then (a) can be replaced by the stronger statement,

(aa) π is K-acyclic.

To say that a map π is K-acyclic means that for each $x \in X$, every map of the fiber $\pi^{-1}(x)$ to K is nullhomotopic.

1. INTRODUCTION

The Edwards-Walsh resolution theorem, the first resolution theorem for cohomological dimension, was proved in [Wa] (see also [Ed]). It states that if X is a metrizable compactum and $\dim_{\mathbb{Z}} X \leq n$ $(n \geq 0)$, then there exists a metrizable compactum Z with $\dim Z \leq n$ and a surjective cell-like map $\pi : Z \to X$. This result, in conjunction with Dranishnikov's work ([Dr1]) showing that in the class of metrizable compacta, $\dim_{\mathbb{Z}}$ is distinct from dim, was a key ingredient for proving that cell-like maps could raise dimension (see [Ru1] for background). For the reader seeking fundamentals on the theory of cohomological dimension, $\dim_G$, the references [Ku], [Dr3], [Dy], and [Sh] could be helpful.

In [Wa] the author made use of the fact that cohomological dimension with respect to the abelian group $\mathbb{Z}$ could be defined in terms of extensions of maps. That is, $\dim_{\mathbb{Z}} X \leq n$ if and only if every map of a closed subspace of X to (a CW-complex of type) $K(\mathbb{Z}, n)$ can be extended to a map of X to $K(\mathbb{Z}, n)$. Indeed (see Theorem 26 of [Ku]), for any abelian group G, $\dim_G X \leq n$ if and only if every map

of a closed subspace of X to $K(G,n)$ can be extended to a map of X to $K(G,n)$. Always by $K(G,n)$ we mean an Eilenberg-Mac Lane CW-complex of type $K(G,n)$, and such is characterized (up to homotopy equivalence) by having $\pi_n = G$ and π_k trivial for all other k.

Motivated in part by this, a concept called "extension theory" was introduced by A. Dranishnikov (see [DD] for some background on this notion). For a CW-complex K and a space X, one writes extdim $X \leq K$ (variously, $X\tau K$, $K \in \mathrm{AE}(X)$, $\dim X \leq K$) if for each closed subspace A of X and map $f : A \to K$, there exists a map $F : X \to K$ such that $F|A = f$. If $K = K(G,n)$, then extdim $X \leq K$ is equivalent to $\dim_G X \leq n$.

Now a map is cell-like provided that each of its fibers is cell-like, or, equivalently, has the shape of a point ([MS1]). Every cell-like compactum has trivial reduced Čech cohomology with respect to any abelian group G. This means that for every abelian group G, every cell-like map is G-acyclic, i.e., all its fibers have trivial reduced Čech cohomology with respect to the group G. This is equivalent to the statement that every map of such a fiber to $K(G,n)$ is nullhomotopic.

The latter notion may be generalized as follows. For a given CW-complex K, a metrizable compactum X is called K-acyclic if every map of it to K is nullhomotopic. Moreover, one should recall that when a Hausdorff compactum or metrizable space X has $\dim X \leq n$, then also $\dim_{\mathbb{Z}} X \leq n$.

With these ideas in mind, one may ask, what kind of parallel resolution theorems can be obtained under the assumption that extdim $X \leq K$, where K is a CW-complex different from $K(\mathbb{Z},n)$? It turns out that it is not possible to always have cell-like resolutions as in the Edwards-Walsh theorem, nor can one even require in such propositions that $\dim Z \leq n$ be true (see [KY2]). So, what kind of resolution theorems can we expect? The main result of this paper is:

1.1. Theorem. *Let K be a connected CW-complex, G an abelian group, $n \in \mathbb{N}_{\geq 2}$, $G = \pi_n(K)$, and $\pi_k(K) = 0$ for $0 \leq k < n$. Then for each metrizable compactum X with* extdim $X \leq K$, *there exists a metrizable compactum Z and a surjective map $\pi : Z \to X$ such that:*

(a) *π is G-acyclic,*
(b) *$\dim Z \leq n + 1$, and*
(c) *extdim $Z \leq K$.*

If in addition, $\pi_{n+1}(K) = 0$, then we may replace (a) *by the stronger statement:*

(aa) *π is K-acyclic.*

As a corollary to Theorem 1.1, we get the G-acyclic resolution theorem in cohomological dimension theory.

1.2. Corollary. *Let G be an abelian group and let X be a metrizable compactum with $\dim_G X \leq n$, $n \in \mathbb{N}_{\geq 2}$. Then there exists a metrizable compactum Z and a surjective map $\pi : Z \to X$ such that:*

(a) *π is G-acyclic,*
(b) *$\dim Z \leq n + 1$, and*
(c) *$\dim_G Z \leq n$.*

In [Le] one finds another approach to Corollary 1.2. We mention that the Edwards-Walsh theorem has been generalized to the class of arbitrary metrizable spaces by Rubin and Schapiro ([RS]) and to the class of arbitrary compact Hausdorff

spaces by Mardešić and Rubin ([MR]). Corollary 1.2 was proved by Dranishnikov ([Dr2]) for the group $G = \mathbb{Z}/p$, where p is an arbitrary prime number, but with the stronger outcome that $\dim Z \leq n$. Later, Koyama and Yokoi ([KY1]) were able to obtain this $\mathbb{Z}/p$-resolution theorem of Dranishnikov both for the class of metrizable spaces and for that of compact Hausdorff spaces.

In their work [KY2], Koyama and Yokoi made a substantial amount of progress in the resolution theory of metrizable compacta, that is, towards proving Corollary 1.2. Their method relies heavily on the existence of Edwards-Walsh resolutions, which had been studied by Dydak and Walsh in [DW], and which had been applied originally, in a rudimentary form, in [Wa]. The definition of an Edwards-Walsh resolution can be found in [KY2], but we shall not use it herein.

To overcome a flaw in the proof of Lemma 4.4 of [DW], Koyama and Yokoi proved the existence of Edwards-Walsh resolutions for some groups G, but under a stronger set of assumptions on G than had been thought necessary in [DW]. It is still not known if these stronger assumptions are needed to insure the existence of the resolutions. Nevertheless, Koyama and Yokoi proved substantial G-acyclic resolution theorems. Let us state two of the important theorems from [KY2] (Theorems 4.9 and 4.12, respectively), which greatly influenced the direction of the work in this paper.

1.3. Theorem. *Corollary* 1.2 *is true for every torsion free abelian group* G.

1.4. Theorem. *Let* G *be an arbitrary abelian group and let* X *be a metrizable compactum with* $\dim_G X \leq n$, $n \in \mathbb{N}_{\geq 2}$. *Then there exists a surjective* G-acyclic map $\pi : Z \to X$ *from a metrizable compactum* Z, *where* $\dim Z \leq n + 2$ *and* $\dim_G Z \leq n + 1$.

In case G is a torsion group, Koyama and Yokoi prove (Theorem 4.11 in [KY2]) that Corollary 1.2 holds, but without part (c). Of course Theorem 1.4 falls short of Corollary 1.2. We observed that one of the main reasons for the relative weakness of this theorem was that Koyama and Yokoi proved it by an indirect technique, a type of "finesse." Their approach depends heavily on the Bockstein basis theorem and the Bockstein inequalities (see [Ku]), instead of the more direct method, involving Edwards-Walsh resolutions, used to prove Theorem 1.3.

We want to point out that Theorem 1.3 includes as a corollary, and therefore redeems, the $\mathbb{Q}$-resolution theorem of Dranishnikov ([Dr5]–but see also [Dr6], where a different proof is given). The Koyama and Yokoi proof shows that in the proof of Theorem 3.2 of [Dr5], the statement that $\alpha_m \circ \omega_m$ is an Edwards-Walsh resolution over $\tau_m^{(n+1)}$ is not true. This was a subtle point; to fully understand it, the interested reader may examine the text immediately following the proof of Fact 1 of the proof of Theorem 3.1 in [KY2]. Getting around the barrier naturally led to a quite complicated construction.

Our proof of Theorem 1.1 will be direct, using extensions which are different from Edwards-Walsh resolutions. But we will use a type of pseudo-Bockstein basis denoted $\sigma_0(G)$ (section 4). This will allow us to deal with the groups $\mathbb{Z}/p^\infty$ as well as the other groups involved. We shall employ the technique of inverse sequences both to represent our given space X and to determine the resolving space Z. The map $\pi : Z \to X$ will be obtained in a standard, yet complicated manner similar to that used in [Wa].

An outline of the sections of this paper goes as follows.

1. Introduction, including some historical facts about acyclic resolutions with respect to certain groups.
2. Eilenberg-Mac Lane complexes $K(G, n)$ and direct sequences of $K(G, n)$'s with an application to $K(\mathbb{Z}/p^\infty, n)$.
3. Some theory of inverse sequences of compacta along with Dranishnikov's lemma for obtaining maps of a compactum to a given compactum with discernable fibers.
4. Some basic algebra needed for this work including a variant Bockstein theory.
5. Extensions of generic type will be defined here. The concept involves starting with a simplicial complex of dimension $\leq n + 1$, removing the interiors of all its $(n + 1)$-simplexes, replacing them with other objects to create a CW-complex, and then defining a map of this CW-complex back to the polyhedron of the original simplicial complex.
6. Layered extensions, certain CW-complexes, are introduced in this section. Using the concepts from section 5, but with a more specific goal in mind, we build extensions in a finite number of layers and then in a reverse process attach $(n + 1)$-cells to improve the homotopy groups of special subspaces of these CW-complexes.
7. Our proof of the G-acyclic resolution theorem involves a certain recursive process. The purpose of this section is to show how to make the move in the induction from stage w to stage $w + 1$.
8. Pulling together the elements of the preceding sections, this ultimate one will provide the final steps in our proof of Theorem 1.1.

2. Lemmas for Eilenberg-Mac Lane complexes

The purpose of this section is to illuminate certain facts about Eilenberg-Mac Lane complexes in a form that will be useful for us. Map will mean continuous function, and for a simplicial complex K, its polyhedron $|K|$ will always be given the weak topology based on the triangulation K.

2.1. Lemma. *Suppose that (G_i, p_{i+1}^i) is a direct sequence of injective homomorphisms $p_{i+1}^i : G_i \to G_{i+1}$ of abelian groups whose limit is the abelian group G. Then for each $n \in \mathbb{N}$, there exists a direct sequence $(K(G_i, n), q_{i+1}^i)$ of $K(G_i, n)$'s and embeddings $q_{i+1}^i : K(G_i, n) \to K(G_{i+1}, n)$, which induce injective homomorphisms on homotopy groups, and whose limit equals $K(G, n)$.*

Proof. Our proof is only for $n \geq 2$; we leave the case $n = 1$ to the reader. Suppose $m \in \mathbb{N}$ and we have constructed CW-complexes $K_1, \ldots, K_m$, and maps $q_{i+1}^i : K_i \to K_{i+1}$ for $i < m$ such that for $1 \leq i \leq m$,

(1) K_i is a $K(G_i, n)$, and
(2) $\pi_n(q_{i+1}^i) : \pi_n(K_i) \to \pi_n(K_{i+1})$ is injective for $i < m$.

Let S be a bouquet of α copies of S^n, where α is the cardinality of a generating set of G_{m+1}. Form the wedge $K_m \vee S$ and note that $\pi_j(K_m \vee S) = 0$ for $j < n$. Also note that by Theorem V.3.1 of [Hu], $\pi_n(K_m \vee S)$ is isomorphic to $G_m \oplus \pi_n(S) \oplus H$. Attach as needed a set of $(n+1)$-cells to $K_m \vee S$ to kill the summand H. This yields a CW-complex L_1 with $\pi_j(L_1) = 0$ for $j < n$ and $\pi_n(L_1) \approx G_m \oplus \pi_n(S)$. Next attach $(n+1)$-cells to L_1 to form a CW-complex L_2 with $\pi_n(L_2) \approx G_m \oplus G_{m+1}$ and $\pi_j(L_2) = 0$ for $j < n$.

Fix a set U of generators of G_m. For each $u \in U$, we consider the element $u - p_{m+1}^m(u) \in \pi_n(L_2)$, and attach an $(n + 1)$-cell to L_2 to kill this element. The resulting CW-complex L_3 has the property that $\pi_j(L_3) = 0$ for $j < n$ and $\pi_n(L_3) \approx G_{m+1}$. Moreover, the homomorphism on π_n induced by the inclusion $K_m \hookrightarrow L_3$ injects G_m into G_{m+1} as a homomorphism equivalent to $p_{m+1}^m : G_m \to G_{m+1}$.

The final step in producing K_{m+1} is to attach cells of dimensions $n+2$ and higher to L_3 as necessary to kill all the higher homotopy groups. From this construction it is clear that K_{m+1} is a $K(G_{m+1}, n)$ and that the inclusion $q_{m+1}^m : K_m \to K_{m+1}$ induces the injective homomorphism $p_{m+1}^m : G_m \to G_{m+1}$. This concludes our proof. $\qquad\square$

We shall apply Lemma 2.1 in the sequel for $p \in \mathbb{P}$ and $G = \mathbb{Z}/p^\infty$. Recall that $\mathbb{Z}/p^\infty$ may be represented as the direct limit of a direct sequence (G_i, p_{i+1}^i), where for each $i \in \mathbb{N}$,

(1) $G_i = \mathbb{Z}/p^i$, and
(2) $p_{i+1}^i : G_i \to G_{i+1}$ is the injective homomorphism induced by multiplication by p.

The next could be used to prove the Bockstein inequality, $\dim_{\mathbb{Z}_{p^\infty}} X \leq \dim_{\mathbb{Z}/p} X$.

2.2.   Corollary. *For each $p \in \mathbb{P}$ and $n \in \mathbb{N}$, there exists a direct sequence $(K(\mathbb{Z}/p^i, n), q_{i+1}^i)$ of embeddings q_{i+1}^i which, on n-th homotopy groups, induce the injective homomorphisms in (2), and such that the limit of the sequence equals $K(\mathbb{Z}/p^\infty, n)$.* $\qquad\square$

In Exercise 5 of [Wa], it is stated that $K(\bigoplus_1^l \mathbb{Z}, n)$ is an Eilenberg-Mac Lane space of type $\prod_1^l K(\mathbb{Z}, n)$ for any $l \in \mathbb{N}$. A similar result is used implicitly in [KY2] and elsewhere for groups G different from $\mathbb{Z}$. On the other hand, as pointed out on page 401 of [Sp] (in reference to a result of C. H. Dowker), the product of two CW-complexes need not be a CW-complex, although it will be if at least one of the factors is locally compact. This brings up the question: what is really needed concerning $K(\bigoplus_1^l G, n)$ in the theory of resolutions in cohomological dimension? Let us give an answer which applies to a large class of spaces.

If K and L are CW-complexes, then by $K \times_{\mathrm{CW}} L$ is meant the set $K \times L$ with the weak topology induced by the CW-structure obtained from the product of the cells of K and L. This topology need not agree with the product topology, but the identity function $i : K \times_{\mathrm{CW}} L \to K \times L$ is continuous. Recall that a k-space is a topological space whose topology is compactly generated. Polyhedra with the weak topology, CW-complexes, first countable spaces, and compact Hausdorff spaces are examples of k-spaces.

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Let us state Proposition 1.5 of [DD] here for the reader's convenience.

2.3. Proposition. *Let X be a k-space and let K, L be CW-complexes. Then a function $f : X \to K \times_{\mathrm{CW}} L$ is continuous if and only if $i \circ f : X \to K \times L$ is continuous.*

2.4. Lemma. *Let $H = \bigoplus_1^l G$ be a finite direct sum of copies of an abelian group G, let X be a paracompact k-space, and let $n \in \mathbb{N}$. Then $\dim_G X \leq n$ if and only if for every closed subset A of X and map $f : A \to K(H, n)$, there exists a map $F : X \to K(H, n)$ which is an extension of f.*

Proof. Using Proposition 2.3, it is not difficult to see that $\pi_k(\prod_{1 \ \mathrm{CW}}^l K(G, n)) \approx \pi_k(\prod_1^l K(G, n))$ for all k. Hence $\prod_{1 \ \mathrm{CW}}^l K(G, n)$ is a $K(H, n)$.

If $\dim_G X \leq n$, then surely $\prod_1^l K(G, n)$ is an absolute extensor for X, since the same is true for each of its factors. Applying Proposition 2.3 again, one concludes that for any closed subset A of X and map $f : A \to K(H, n)$, there is a map $F : X \to K(H, n)$ extending f.

For the converse, let $A \subset X$ be closed and $f : A \to K(G, n)$ be a map. Denote by $f^* : A \to \prod_{1 \ \mathrm{CW}}^l K(G, n) = K(H, n)$ the function given by $f^*(a) = (f(a), \ldots, f(a))$. Using Proposition 2.3, we see that f^* is continuous, so by assumption it extends to a map $F^* : X \to \prod_{1 \ \mathrm{CW}}^l K(G, n)$. Now Proposition 2.3 shows that $i \circ F^* : X \to \prod_1^l K(G, n)$ is continuous. The map $F = p_1 \circ i \circ F^* : X \to K(G, n)$, where p_1 is projection to the first coordinate, is an extension of f. $\square$

2.5. Lemma. *Let $H = \bigoplus_1^l G$ be a finite direct sum of copies of an abelian group G, let X be a metrizable space, and let $n \in \mathbb{N}$. Assume that $\dim_G X \leq n$, $\dim X \leq n + 1$, and K is a CW-complex having the property that*

$$\pi_k(K) \approx \begin{cases} 0 & \text{if } k < n, \\ H & \text{if } k = n. \end{cases}$$

Then $\operatorname{extdim} X \leq K$.

Proof. Surely, $K \subset K(H, n)$, where $K(H, n) \backslash K$ consists of cells of dimensions k, $k \geq n + 2$. If $A \subset X$ and $f : A \to K$ is a map, then by Lemma 2.4 there exists a map $F : X \to K(H, n)$ extending f. Dimensional considerations and the homotopy extension theorem show that we may assume that $F(X) \subset K(H, n)^{(n+1)} = K$. $\square$

3. Lemmas for inverse sequences

Our proof of the main result will require certain manipulations of inverse sequences of metric compacta. This section will contain the needed results.

The next lemma follows from Corollary 1 of [MS2].

3.1. Lemma. *Let* $\mathbf{X} = (X_i, p_i^{i+1})$ *be an inverse sequence of metric compacta* (X_i, d_i). *Then there exists a sequence* (δ_i) *of positive numbers such that if* $\mathbf{Y} = (X_i, q_i^{i+1})$ *is an inverse sequence and* $d_i(p_i^{i+1}, q_i^{i+1}) < \delta_i$ *for each* i, *then* $\lim \mathbf{Y} = \lim \mathbf{X}$. $\qquad\square$

We shall call such (δ_i) a **sequence of stability** for $\mathbf{X}$.

Henceforth if we write that $f : |L| \to |K|$ is a simplicial map, we mean that f is induced by a simplicial function $L \to K$. As usual, when $f : L \to K$ is a simplicial map of simplicial complexes, we shall use $f : |L| \to |K|$ to denote the induced map of polyhedra. Let us recall some terminology from section 2 of [JR]. Let K be a finite simplicial complex, let X be a space, and let $f : X \to |K|$ be a map. Then a map $g : X \to |K|$ is called a K-*modification* of f if whenever $x \in X$ and $f(x) \in \sigma \in K$, then $g(x) \in \sigma$. It should be observed that if L is a simplicial complex, $X = |L|$, and $g : |L| \to |K|$ is a simplicial approximation to f, then g is a K-modification of f. One calls f a K-*irreducible* map if each K-modification g of f is surjective. Note that in this case, f is surjective and for any subdivision M of K, f is M-irreducible.

From Theorem 3.11 of [JR] we may deduce the following.

3.2. Lemma. *Let* X *be a metrizable compactum. Then we may write* X *as the limit of an inverse sequence* $\mathbf{Q} = (|Q_i|, q_i^{i+1})$ *of compact metric polyhedra, where each bonding map* q_i^{i+1} *is a* Q_i-*irreducible surjection. If* $\operatorname{card} X \geq 2$, *then we may assume that* $\operatorname{card} Q_i \geq 2$ *for each* $i \in \mathbb{N}$. $\qquad\square$

Let us recall that by an *interior cover* of a space X we mean a cover of X by sets whose interiors also cover X. We need Dranishnikov's definition of a certain type of Lebesgue number.

Let X be a compact metric space and let $\mathcal{M}$ be an interior cover of X by closed sets such that no member of $\mathcal{M}$ equals X. Then there is a positive number $\lambda^u(\mathcal{M})$ given by

$$\lambda^u(\mathcal{M}) = \max\{r \,|\, \forall x \in X \,\exists M \in \mathcal{M}, B(x, r) \subset M\}.$$

The next lemma is clearly true.

3.3. Lemma. *If* X *is a nonempty compact metric space and* $\mathcal{M}$ *is an interior cover of* X *by closed sets such that no member of* $\mathcal{M}$ *equals* X, *then* $\lambda^u(\mathcal{M})/4 < \lambda^u(\mathcal{M}) \leq \operatorname{mesh} \mathcal{M}$.

For a given simplicial complex K, let $\overline{\operatorname{st}}K$ denote the set of closed vertex stars of K. Surely $\overline{\operatorname{st}}K$ is an interior cover of $|K|$ by closed sets.

3.4. Lemma. *Let* X *be a metrizable compactum with* $\operatorname{card} X \geq 2$. *Then there exists an inverse sequence* $\mathbf{K} = (|K_i|, p_i^{i+1})$ *of compact metric polyhedra* $(|K_i|, d_i)$ *along with a sequence* (δ_i) *of stability for* $\mathbf{K}$ *such that* $\lim \mathbf{K} = X$, *and for each* $i \in \mathbb{N}$,

(a) $\operatorname{mesh} K_i < \delta_i$,

(b) *no member of* $\overline{\operatorname{st}}K_i$ *equals* $|K_i|$, *and*

(c) $\operatorname{mesh}(\overline{\operatorname{st}}K_{i+1}) < \lambda^u(\overline{\operatorname{st}}K_i)/4$.

We may also specify that for some $m \in \mathbb{N}$, *whenever* $i \geq m$,

(d) $p_i^{i+1} : |K_{i+1}| \to |K_i|$ *is a* K_i-*irreducible simplicial map.*

Proof. Write $X = \lim \mathbf{Q}$ of compact metric polyhedra (Q_i, d_i) with $\operatorname{card} Q_i \geq 2$ for each $i \in \mathbb{N}$ as in Lemma 3.2, and let (ρ_i) be a sequence of stability for $\mathbf{Q}$. For each i, put $\delta_i = \rho_i/2$.

Below when we select a subdivision K_i of Q_i, we shall tacitly (without loss of generality since $\operatorname{card} Q_i \geq 2$ for each $i \in \mathbb{N}$) assume that no element of $\overline{\operatorname{st}} K_i$ equals $|K_i|$. Let K_1 be a subdivision of Q_1 with $\operatorname{mesh} K_1 < \delta_1$. Suppose that $i \in \mathbb{N}$ and for each $1 \leq j \leq i$, we have chosen a subdivision K_j of Q_j with $\operatorname{mesh} K_j < \delta_j$ and, when $1 < j$, a map $p_{j-1}^j : |K_j| \to |K_{j-1}|$ which is a simplicial approximation to q_{j-1}^j. Assume moreover that when $j < i$, $\operatorname{mesh}(\overline{\operatorname{st}} K_{j+1}) < \lambda^u(\overline{\operatorname{st}} K_j)/4$. Then select a subdivision K_{i+1} of Q_{i+1} with $\operatorname{mesh} K_{i+1} < \delta_{i+1}$, $\operatorname{mesh}(\overline{\operatorname{st}} K_{i+1}) < \lambda^u(\overline{\operatorname{st}} K_i)/4$, and which supports a simplicial approximation $p_i^{i+1} : |K_{i+1}| \to |K_i|$ of q_i^{i+1}.

Note that $d_i(q_i^{i+1}, p_i^{i+1}) < \delta_i$. Then it is easy to check that with $\mathbf{K} = (|K_i|, p_i^{i+1})$ and $m = 1$, we have satisfied all the requirements. $\qquad \square$

3.5. Lemma. *When an inverse sequence $\mathbf{K}$ of metric polyhedra $(|K_i|, d_i)$ satisfies (c) of Lemma 3.4, then for each $i < j$,*

$$\operatorname{mesh}(\overline{\operatorname{st}} K_j) < \lambda^u(\overline{\operatorname{st}} K_i)/4.$$

Proof. Just apply Lemma 3.3 recursively. $\qquad \square$

3.6. Definition. Whenever X is a metrizable compactum, then we shall refer to an inverse sequence $\mathbf{K}$ of metric polyhedra $(|K_i|, d_i)$ which admits a sequence (δ_i) of positive numbers and $m \in \mathbb{N}$ so that (a)–(d) of Lemma 3.4 are satisfied as a representation of X which is **stable and simplicially irreducible** from **index** m.

Of course Lemma 3.4 and its proof show that,

3.7. Lemma. *Every metrizable compactum X with $\operatorname{card} X \geq 2$ has a representation $\mathbf{K}$ which is stable and simplicially irreducible from index $m = 1$.* $\qquad \square$

We want next to define a certain type of move which when applied to such $\mathbf{K} = \mathbf{K}_0$ as in Definition 3.6 results in a $\mathbf{K}_1$ which is also a stable and simplicially irreducible (from some index m) representation of X. We will then show that if this procedure is repeated recursively in a controlled manner, resulting in a sequence $\mathbf{K}_1, \mathbf{K}_2, \dots$, then there will be a limit $\mathbf{K}_\infty = \lim_{j \to \infty} (\mathbf{K}_j)$ which also will be a representation of X.

3.8. Definition. Let (ϵ_i) be a sequence of positive numbers, let X be a metrizable compactum, and let $\mathbf{K} = (|K_i|, p_i^{i+1})$ be a representation of X which is stable and simplicially irreducible from index m_1 with (δ_i) an associated sequence of stability. Let $m \in \mathbb{N}_{\geq m_1}$ and define $\delta_i' = \delta_i$ if $1 \leq i < m$, $\delta_m' = \frac{1}{2}[\delta_m - \operatorname{mesh} K_m]$, and $\delta_i' = \delta_i/2$ if $i > m$. There exists a sequence $\Sigma = L_m, L_{m+1}, L_{m+2}, \dots$ such that for each $i \geq m$,

(a) L_i is a subdivision of K_i with $\operatorname{mesh} L_i < \min\{\varepsilon_i, \delta_i'\}$, and
(b) if $i \geq m + 1$ there is a simplicial approximation $l_{i-1}^i : |L_i| \to |L_{i-1}|$ to the map p_{i-1}^i.

Create a new inverse sequence $\mathbf{L} = (|L_i|, l_i^{i+1})$ by setting, in addition to what we have just described, $L_i = K_i$ and $l_i^{i+1} = p_i^{i+1}$ in case $1 \leq i < m$. Observe that $d_m(p_m^{m+1}, l_m^{m+1}) \leq \operatorname{mesh} \Sigma < \frac{1}{2}[\delta_m - \operatorname{mesh} K_m] = \delta_m'$. Hence if $g : |L_{m+1}| \to |L_m|$

is a map and $d_m(g, l_m^{m+1}) < \delta_m'$, we may conclude that $d_m(g, p_m^{m+1}) < \delta_m$. Indeed, and we leave it to the reader to check, the following is true:

   (c) for each i, if $g : |L_{i+1}| \to |L_i|$ is a map and $d_i(g, l_i^{i+1}) < \delta_i'$, then $d_i(g, p_i^{i+1}) < \delta_i$.

By exercising some additional care in the construction of $\mathbf{L}$, we may guarantee that (c) of Lemma 3.4 applies to it, and

   (d) for all i, $d_i(p_i^{i+1}, l_i^{i+1}) < \epsilon_i$ and, of course, $p_i^{i+1} = l_i^{i+1}$ if $i < m$.

We shall call such $(\mathbf{L}, (\delta_i'))$ an m-**shift** of $(\mathbf{K}, (\delta_i))$ from Σ.

   It is routine to check that the next lemma holds true.

3.9. Lemma. *Let X be a metrizable compactum and let $\mathbf{K}_0$ be a stably irreducible representation of X with (δ_i^0) a sequence of stability and which is simplicial from index m_1. For every m_1-shift $(\mathbf{K}_1, (\delta_i^1))$ of $(\mathbf{K}_0, (\delta_i^0))$ from Σ_1 (an appropriate subdivision of the triangulation of the m_1-term of $\mathbf{K}_0$), $\mathbf{K}_1$ is a representation of X which is stable and simplicially irreducible from index m_1 with (δ_i^1) an associated sequence of stability. It satisfies (c) of Definition 3.8 with $(\delta_i') = (\delta_i^1)$ and $(\delta_i) = (\delta_i^0)$. The terms (as metric spaces) in $\mathbf{K}_0$ and $\mathbf{K}_1$ are equal. For $i < m_1$, $\delta_i^0 = \delta_i^1$, the terms with index i have the same triangulations in $\mathbf{K}_0$ and $\mathbf{K}_1$, and the bonding maps in $\mathbf{K}_0$ and $\mathbf{K}_1$ with subscript i are equal. For $i \geq m_1$, δ_i^1 need not equal δ_i^0, the triangulation of the term in $\mathbf{K}_1$ with index i is a subdivision of that in $\mathbf{K}_0$ with the same index, and the bonding map with subscript i in $\mathbf{K}_1$ may differ from that in $\mathbf{K}_0$ with subscript i.*

If $i_0 \in \mathbb{N}$, $m_1 < \cdots < m_{i_0}$ is a finite sequence in $\mathbb{N}$, and successively we have chosen $(\mathbf{K}_j, (\delta_i^j))$ an m_j-shift of $(\mathbf{K}_{j-1}, (\delta_i^{j-1}))$ from Σ_j (an appropriate subdivision of the m_j-term of $\mathbf{K}_{j-1}$), $1 \leq j \leq i_0$, then we may conclude that $\mathbf{K}_{i_0}$ is a representation of X which is stable and simplicially irreducible from index m_{i_0} with $(\delta_i^{i_0})$ an associated sequence of stability; it satisfies (c) of Definition 3.8 with $(\delta_i') = (\delta_i^{i_0})$ and $(\delta_i) = (\delta_i^0)$. The terms (as metric spaces) in $\mathbf{K}_0$ and $\mathbf{K}_{i_0}$ are equal. For $i < m_{i_0}$, $\delta_i^{i_0-1} = \delta_i^{i_0}$, the terms with index i have the same triangulations in $\mathbf{K}_{i_0-1}$ and $\mathbf{K}_{i_0}$, and the bonding maps in $\mathbf{K}_{i_0-1}$ and $\mathbf{K}_{i_0}$ with subscript i are equal. For $i \geq m_{i_0}$, $\delta_i^{i_0}$ need not equal $\delta_i^{i_0-1}$, the triangulation of the term in $\mathbf{K}_{i_0}$ with index i is a subdivision of that in $\mathbf{K}_{i_0-1}$ with the same index, and the bonding map with subscript i in $\mathbf{K}_{i_0}$ may differ from that in $\mathbf{K}_{i_0-1}$ with subscript i.

In consideration of (d) of Definition 3.8, we may also conclude that if we are given a sequence (ϵ_i) of positive numbers, then for each $i \in \mathbb{N}$,

$$d_i(p_i^{i+1}, q_i^{i+1}) < \epsilon_i,$$

where q_i^{i+1} denotes the appropriate bonding map from $\mathbf{K}_{i_0}$, p_i^{i+1} the one from $\mathbf{K}_0$.

$\square$

   Henceforth we typically shall write $(|K_{(j)\,i}|, p_{(j)\,i}^{i+1})$ to denote such a representation $\mathbf{K}_j$, $0 \leq j \leq i_0$. One should note that, whenever $i_0 \geq j_0 \geq j \geq 1$, then $K_{(j)\,m_j} = K_{(j_0)m_j} = \Sigma_j$ when this occurs from the procedure in Lemma 3.9.

3.10. Definition. Let X be a metrizable compactum and let $r : \mathbb{N} \to \mathbb{N}$ be an increasing function. Let $\mathbf{K}_0$ be a representation of X which is stable and simplicially irreducible from index $r(1)$ with (δ_i^0) a sequence of stability. Suppose that $(\mathbf{K}_j, (\delta_i^j))$, $j \in \mathbb{N}$, is a sequence such that for each j, $(\mathbf{K}_j, (\delta_i^j))$ is an $r(j)$-shift of $(\mathbf{K}_{j-1}, (\delta_i^{j-1}))$ from Σ_j.

Then for each $k \in \mathbb{N}$, if m, l, and i are chosen so that $m \geq l \geq r(k) > i$, one sees that $p^{i+1}_{(l)\,i} = p^{i+1}_{(m)\,i}$ and $\delta^l_i = \delta^m_i$. So for each i, the sequences (δ^j_i) and $(p^{i+1}_{(j)\,i})$ are eventually constant. Hence, in an obvious way, we may define an inverse sequence $\mathbf{K}_\infty = (|K_{(\infty)\,i}|, p^{i+1}_{(\infty)\,i}) = \lim_{j \to \infty} \mathbf{K}_j$, and a sequence $(\delta^\infty_i) = \lim_{j \to \infty}(\delta^j_i)$ of positive numbers. Here, $K_{(\infty)\,i} = \lim_{j \to \infty} K_{(j)\,i}$ and $p^{i+1}_{(\infty)\,i} = \lim_{j \to \infty} p^{i+1}_{(j)\,i}$.

The following is not difficult to deduce from our construction and this definition.

3.11. Lemma. *Assume the notation of Definition* 3.10. *Then* $\mathbf{K}_\infty$ *is a representation of* X. *If* $i \in \mathbb{N}$, $g : |K_{(\infty)\,i+1}| \to |K_{(\infty)\,i}|$ *is a map, and* $d_i(g, p^{i+1}_{(\infty)\,i}) < \delta^\infty_i$, *then* $d_i(g, p^{i+1}_{(0)\,i}) < \delta^0_i$ *and hence* (δ^∞_i) *is a sequence of stability for* $\mathbf{K}_\infty$.

Next is a well-known result which follows from the fact that if H is a countable abelian group, then we may (and always do) express $K(H, n)$ as a countable CW-complex.

3.12. Lemma. *Let* X *be a compact metrizable space,* H *a countable abelian group, and* $n \in \mathbb{N}$. *Then there exists a countable set* $\mathcal{H}(X, H)$ *of maps of closed subspaces of* X *to* $K(H, n)$ *such that for each closed subspace* B *of* X *and map* $h : B \to K(H, n)$, *there is* $h_0 : B_0 \to K(H, n)$ *in* $\mathcal{H}(X, H)$, *where* $B \subset B_0$ *and* $h_0|B \simeq h$.

Recall (see the Introduction) that for each abelian group H and metrizable compactum X, $\operatorname{extdim} X \leq K(H, n)$ is equivalent to the statement $\operatorname{extdim}_H X \leq n$. Applying Theorem 1.4 of [Ru2], we may state,

3.13. Lemma. *Let* H *be a countable abelian group,* $n \in \mathbb{N}$, $\mathbf{V} = (V_i, g^{i+1}_i)$ *an inverse sequence of metrizable compacta, and* $V = \lim \mathbf{V}$. *Suppose that for each* i, $\mathcal{H}(V_i, H)$ *as in Lemma* 3.12 *has been chosen and for each* $h : A \to K(H, n)$ *in* $\mathcal{H}(V_i, H)$, *there exists* $k \geq i$ *and a map* $g : V_k \to K(H, n)$ *such that for each* $x \in (g^k_i)^{-1}(A)$, $h(g^k_i(x)) = g(x)$. *Then* $\operatorname{extdim} V \leq K(H, n)$ *and hence* $\dim_H V \leq n$.

In [Wa] the author developed a technique, using an inverse sequence of polyhedra whose limit is a given metrizable compactum X, of finding a metrizable compactum Z and a map $\pi : Z \to X$. Along with this, he showed how to recognize the fibers of π as inverse limits of sub-inverse sequences of an inverse sequence whose limit is Z. This concept has been refined, and will be quite useful to us in its evolved form. We explain this idea, due to A. Dranishnikov ([Dr4]), as follows.

Let $(\tilde{K}_i, d_i)$ be a sequence of compact metric spaces having the property that for each i, $\operatorname{diam}_i \tilde{K}_i < 2^{-i}$. Under these conditions, we shall always use the brick metric ρ on $\prod_{i=1}^\infty \tilde{K}_i$ given by

$$\rho((x_i), (y_i)) = \sum_{i=1}^\infty d_i(x_i, y_i).$$

In section 3 of [Dr4] the author uses this along with the notion of a realization of an inverse sequence $\mathbf{X} = (\tilde{K}_i, \tilde{p}^{i+1}_i)$ to obtain an important lemma. Now we state Lemma 4 of [Dr4].

3.14. Lemma. *Let* $\mathbf{X} = (\tilde{K}_i, \tilde{p}^{i+1}_i)$ *and* $\mathbf{Z} = (M_i, g^{i+1}_i)$ *be inverse sequences of nonempty metric compacta,* $X = \lim \mathbf{X}$ *and* $Z = \lim \mathbf{Z}$. *Assume that for each* $i \in \mathbb{N}$, *a closed interior cover* $\tilde{\mathcal{M}}_i$ *of* $\tilde{K}_i$ *is given with the additional property that*

no element of $\tilde{\mathcal{M}}_i$ equals $\tilde{K}_i$ and that for each $x \in \tilde{K}_i$, we have selected an element $M_{x,i} \in \tilde{\mathcal{M}}_i$ having the property that $B(x, \lambda^u(\tilde{\mathcal{M}}_i)) \subset M_{x,i}$. Suppose also that a map $\phi_i : M_i \to \tilde{K}_i$ has been chosen such that:

(a) $\phi_i^{-1}(M) \neq \emptyset$ for each $M \in \tilde{\mathcal{M}}_i$,
(b) $d(\tilde{p}_i^{i+1} \circ \phi_{i+1}, \phi_i \circ g_i^{i+1}) < \lambda^u(\tilde{\mathcal{M}}_i)/4$, and
(c) mesh $\tilde{\mathcal{M}}_i < \lambda^u(\tilde{\mathcal{M}}_{i-1})/4$ when $i > 1$.

Then for each $x = (x_i) \in X$, $g_i^{i+1}(\phi_{i+1}^{-1}(M_{x_{i+1},i+1})) \subset \phi_i^{-1}(M_{x_i,i})$, and there is a surjective map $\pi : Z \to X$ such that $\pi^{-1}(x) = \lim \mathbf{Z}_x$, where

$$\mathbf{Z}_x = (\phi_i^{-1}(M_{x_i,i}), g_i^{i+1}). \qquad \square$$

4. Some lemmas from algebra

The following lemma was motivated by the proof of Fact 2 of the proof of Theorem 3.1 of [KY2].

4.1. Lemma. Let G be an abelian group, let $k \in \mathbb{N}$, and let $u \in G$ be divisible by $q \in \mathbb{N}_{\geq 2}$. Suppose that $r \geq 1$ and $\Delta : \mathbb{Z} \to \mathbb{Z} \oplus (\bigoplus_1^r G)$ is the homomorphism induced by $\Delta(1) = (-q, u, -u, -u, \ldots, -u)$. Then

$$[\mathbb{Z} \oplus (\bigoplus_1^r G)]/\operatorname{im} \Delta \approx \mathbb{Z}/q \oplus (\bigoplus_1^r G).$$

Proof. Choose a $z \in G$ such that $qz = u$. Define $h : \mathbb{Z} \oplus (\bigoplus_1^r G) \to \mathbb{Z}/q \oplus (\bigoplus_1^r G)$ by

$$h(i, x_1, \ldots, x_r) = ([i]_q, x_1 + iz, x_2 + x_1, \ldots, x_r + x_1).$$

We see that h is a homomorphism because

$$h(i, x_1, \ldots, x_r) + h(j, y_1, \ldots, y_r)$$
$$= ([i]_q, x_1 + iz, x_2 + x_1, \ldots, x_r + x_1) + ([j]_q, y_1 + jz, y_2 + y_1, \ldots, y_r + y_1)$$
$$= ([i+j]_q, x_1 + y_1 + (i+j)z, x_2 + y_2 + (x_1 + y_1), \ldots, x_r + y_r + (x_1 + y_1))$$
$$= h((i+j, x_1 + y_1, \ldots, x_r + y_r).$$

Next, we show that h is a surjection. Let $(a_0, a_1, \ldots, a_r) \in \mathbb{Z}/q \oplus (\bigoplus_1^r G)$. Thus $a_0 = [i]_q$ for some i. Put $x_1 = a_1 - iz$, and $x_k = a_k - x_1$ for $2 \leq k \leq r$. Then $h(i, x_1, \ldots, x_r) = ([i]_q, a_1 - iz + iz, a_2, \ldots, a_r) = (a_0, a_1, \ldots, a_r)$ as needed.

Now we will detect that h induces an isomorphism as desired by this lemma. To see this, first note that $\operatorname{im} \Delta \subset \ker h$ because $h(\Delta(1)) = h(-q, u, -u, \ldots, -u) = ([-q]_q, u - qz, u - u, \ldots, u - u) = (0, u - u, 0, \ldots, 0) = (0, 0, \ldots, 0)$. On the other hand, $\ker h \subset \operatorname{im} \Delta$ for the following reason. Suppose $h(i, x_1, x_2, \ldots, x_r) = ([i]_q, x_1 + iz, x_2 + x_1, \ldots, x_r + x_1) = 0$. Then $i = qs$ for some s. Also, $x_1 + iz = x_1 + qsz = x_1 + su = 0$, showing that $x_1 = -su$. For $2 \leq k \leq r$, $0 = x_k + x_1 = x_k - su$, i.e., $x_k = su$. Finally, $\Delta(-s) = (-q(-s), -su, su, \ldots, su) = (i, x_1, x_2, \ldots, x_r)$. $\square$

Next are some facts from the theory of abelian groups.

4.2. Lemma. Let $p \in \mathbb{P}$, $q \in \mathbb{N}_{\geq 2}$ and $u \in \mathbb{Z}/p^\infty$. Then u is divisible by q. $\square$

4.3. Corollary. *Let $p \in \mathbb{P}$, $q \in \mathbb{N}_{\geq 2}$, $r \in \mathbb{N}$, $G = \bigoplus_1^r \mathbb{Z}/p^\infty$, and $u \in G$. Then u is divisible by q.* □

Let P_0 be a subset of $\mathbb{P}$. Recall that the group $\mathbb{Z}_{(P_0)}$ is the set of all elements b of $\mathbb{Q}$ such that when $b = \frac{r}{s}$ reduced to lowest terms, then s has no factors from P_0.

4.4. Lemma. *Let $P_0 \subset \mathbb{P}$, let $u \in \mathbb{Z}_{(P_0)}$, and let q be a composite of elements of $\mathbb{P} \setminus P_0$. Then u is divisible by q.* □

From [KY2], we have the following fact.

4.5. Lemma. *Let $P_0 \subset \mathbb{P}$. Then $G = \mathbb{Z}_{(P_0)}$ satisfies the conditions (EW_1) and (EW_2) of [KY2]. In particular, there exists a homomorphism $\alpha : \mathbb{Z} \to G$ such that*

$$\alpha^* : Hom(G, G) \to Hom(\mathbb{Z}, G)$$

is a surjection. □

In order to prove Theorem 1.1 we shall need to form a resolution of the given compactum X in terms of a certain countable set of groups. To detect which groups are needed, we shall apply some Bockstein theory. A good source for this is §2 of [Dr6]; it can also be found in [Ku], but with different notation. For an abelian group G, we shall be involved with its Bockstein basis $\sigma(G)$ in an indirect manner. Henceforward we shall denote by P_G the set of all $p \in \mathbb{P}$ such that $\mathbb{Z}_{(p)} \in \sigma(G)$.

4.6. Lemma. *Let G be an abelian group, $n \in \mathbb{N}$, and $J = \mathbb{Z}_{(P_G)}$. If X is a compact metrizable space and $\dim_G X \leq n$, then $\dim_J X \leq n$, $\dim_{\mathbb{Z}_{(p)}} X \leq \dim_J X$ for all $p \in P_G$, and for some $p^* \in P_G$, $\dim_{\mathbb{Z}_{(p^*)}} X = \dim_J X$.*

Proof. Let $p \in P_G$. Since $\mathbb{Z}_{(p)} \in \sigma(G)$, then $\dim_{\mathbb{Z}_{(p)}} X \leq \dim_G X \leq n$. On the other hand, $\sigma(J) = \{\mathbb{Z}_{(p)} \mid p \in P_G\}$. Hence $\dim_{\mathbb{Z}_{(p)}} X \leq \dim_J X$ for all $p \in P_G$, and $\dim_J X = \sup\{\dim_{\mathbb{Z}_{(p)}} X \mid p \in P_G\} \leq n$. Thus for some $p^* \in P_G$, $\dim_J X = \dim_{\mathbb{Z}_{(p^*)}} X \leq n$. □

4.7. Definition. Let G be an abelian group. We define a set $\sigma_0(G)$ of abelian groups H by requiring that $H \in \sigma_0(G)$ if:

    (a) $H = \mathbb{Z}_{(P_G)}$,
    (b) $H = \mathbb{Z}/p^\infty$, $\mathbb{Z}/p^\infty \in \sigma(G)$, $\mathbb{Z}/p \notin \sigma(G)$, and $p \notin P_G$, or
    (c) $H = \mathbb{Z}/p$, $\mathbb{Z}/p \in \sigma(G)$, and $p \notin P_G$.

4.8. Lemma. *Let G be an abelian group, let $H \in \sigma_0(G)$ be as in (a) or (b) of Definition 4.7, let $q \in \mathbb{N}$ be a composite of primes not in P_G, and let $u \in H$. Then u is divisible by q.*

Proof. This follows from Lemmas 4.4 and 4.2. □

4.9. Lemma. *Let X be a metrizable compactum, let G be an abelian group, let $\sigma_0(G)$ be as in Definition 4.7, and let $n \in \mathbb{N}$. Then $\dim_G X \leq n$ if and only if $\dim_H X \leq n$ for all $H \in \sigma_0(G)$.*

Proof. Suppose $\dim_G X \leq n$. Let $H \in \sigma_0(G)$. If $H \in \sigma(G)$, then of course the Bockstein basis theorem shows that $\dim_H X \leq n$. If $H \notin \sigma(G)$, then H is of the form $\mathbb{Z}_{(P_G)}$ (see (a) of Definition 4.7). Apply Lemma 4.6 to conclude that $\dim_H X \leq n$.

For the opposite implication, suppose $\dim_H X \leq n$ for all $H \in \sigma_0(G)$. Let $H' \in \sigma(G) \backslash \sigma_0(G)$. We simply have to show that $\dim_{H'} X \leq n$ (for all such H').

If for some $p \in \mathbb{P}$, $H' = \mathbb{Z}_{(p)}$, then (a) of Definition 4.7 and Lemma 4.6 prevail. Suppose $H' = \mathbb{Q}$. If $P_G = \emptyset$, then $\mathbb{Q} \in \sigma_0(G)$ by Definition 4.7(a); hence it must be true that $P_G \neq \emptyset$. Pick $p \in P_G$. Then Lemma 4.6 shows that $\dim_{\mathbb{Z}_{(p)}} X \leq \dim_{\mathbb{Z}_{(P_G)}} X \leq n$. The Bockstein inequality BI4 of [Dr6] yields $\dim_{\mathbb{Q}} X \leq n$.

Now consider the case $H' = \mathbb{Z}/p^\infty$ and $p \notin P_G$. Since $\mathbb{Z}/p^\infty \notin \sigma_0(G)$, parts (b) and (c) of Definition 4.7 show that $\mathbb{Z}/p \in \sigma_0(G)$. Hence $\dim_{\mathbb{Z}/p} X \leq n$ by assumption. But the Bockstein inequality BI1 in [Dr6] tells us that $\dim_{H'} X \leq \dim_{\mathbb{Z}/p} X \leq n$. If $p \in P_G$, then the Bockstein inequalities BI1 and BI3 along with Lemma 4.6 indicate that $\dim_{H'} X \leq \dim_{\mathbb{Z}_{(p)}} X \leq \dim_{\mathbb{Z}_{(P_G)}} X \leq n$.

If for some $p \in \mathbb{P}$, $H' = \mathbb{Z}/p$, then because of (c) of Definition 4.7, it must be true that $p \in P_G$. Apply Bockstein inequality BI3 and Lemma 4.6 again. $\square$

For a given abelian group G and metrizable continuum B, we are later going to face the problem of establishing, in terms of the groups in $\sigma_0(G)$, whether B is G-acyclic. The following lemma provides a usable criterion for this.

4.10. Lemma. *Given a metrizable continuum B, abelian group G, and $n \in \mathbb{N}_{\geq 2}$ such that:*

   (a) $\dim_G B \leq n$,
   (b) $\check{H}^k(B; H) = 0$ *for* $1 \leq k < n$ *and any abelian group H,*
   (c) $\check{H}^n(B; H) = 0$ *for all* $H \in \sigma_0(G) \backslash \{\mathbb{Z}_{(P_G)}\}$, *and*
   (d) $\check{H}^n(B; H) = 0$ *for any torsion free abelian group H.*

Then B is G-acyclic.

Proof. We are going to show that B is H-acyclic for all $H \in \sigma(G)$. The result will thus be true by Corollary 4.7 of [KY2]. By (a) and the Bockstein basis theorem, $\dim_H B \leq n$. An application of Theorem 1 of [Ku] shows that

$$\check{H}^{n+k}(B; H) = 0 \text{ for all } k \geq 1.$$

By this and (b) of the hypothesis, we only need to prove that $\check{H}^n(B; H) = 0$.

Suppose that $p \in \mathbb{P}$ and $H \in \{\mathbb{Z}/p, \mathbb{Z}/p^\infty\}$. By (d), $\check{H}^n(B; \mathbb{Z}_{(p)})$ is trivial since $\mathbb{Z}_{(p)}$ is torsion free. By (b), $\check{H}^k(B; \mathbb{Z}_{(p)})$ is trivial for $1 \leq k < n$. If $p \in P_G$, then $\mathbb{Z}_{(p)} \in \sigma(G)$, so we already know that $\check{H}^{n+k}(B; \mathbb{Z}_{(p)}) = 0$ for all $k \geq 1$.

Consider the short exact sequence,

$$0 \to \mathbb{Z}_{(p)} \overset{\times p}{\to} \mathbb{Z}_{(p)} \to \mathbb{Z}/p \to 0.$$

(See [Ku], near the bottom of page 10.) This leads to a long exact sequence in the Čech cohomology of B showing that $\check{H}^q(B; \mathbb{Z}/p) = 0$ for all $q > 0$. Thus, B is $\mathbb{Z}/p$-acyclic. By Lemma 4.2 of [KY2], B is also $\mathbb{Z}/p^\infty$-acyclic. Hence B is H-acyclic in this situation.

On the other hand, suppose that $p \notin P_G$ and consider first the case that $H = \mathbb{Z}/p$. Then by Definition 4.7(c), $\mathbb{Z}/p \in \sigma_0(G)$, so $\check{H}^n(B; H) = 0$ because of (c) of the hypothesis. This of course shows that B is $\mathbb{Z}/p$-acyclic. If $H = \mathbb{Z}/p^\infty$ and $\mathbb{Z}/p \in \sigma(G)$, then once more Lemma 4.2 of [KY2] shows that B is H-acyclic. So we need to concern ourselves with the case that $\mathbb{Z}/p \notin \sigma(G)$ and $H = \mathbb{Z}/p^\infty \in \sigma(G)$. But then Definition 4.7(b) shows that $\mathbb{Z}/p^\infty \in \sigma_0(G)$, so again (c) applies and we see that $\check{H}^n(B; H) = 0$.

The last case to consider is that in which $H \in \sigma(G)$ is torsion free; this is covered by (d) of the hypothesis. $\qquad \square$

4.11. Lemma. *Suppose that G is an abelian group, K is a connected CW-complex, $n \in \mathbb{N}$, $\pi_k(K) = 0$ for $0 \le k < n$, and $\pi_n(K) = G$. Let Z be a metrizable compactum such that $\dim Z \le n + 1$ and $\dim_G Z \le n$. Then $\operatorname{extdim} Z \le K$.*

Proof. We may treat $K = K(G, n)^{(n+1)} \subset K(G, n)$. If $A \subset Z$ is closed and $f : A \to K$ is a map, then $\dim_G Z \le n$ implies that f extends to a map $F : Z \to K(G, n)$. The map F may be replaced by a map F^* that factors through a polyhedron $|L|$ of dimension $\le n+1$ in such a manner that $F^*(A) \subset K$ and F and F^* are homotopic when restricted to maps of A into K. Now use a cellular approximation to adjust the map F^* so that $F^*(Z)$ lands in the $(n + 1)$-skeleton of $K(G, n)$, i.e., in K. Finally apply the homotopy extension theorem to see that f extends to a map of Z into K. $\qquad \square$

The next lemma may be proved in a manner similar to that of Lemma 4.11.

4.12. Lemma. *Suppose that G is an abelian group, K is a connected CW-complex, $n \in \mathbb{N}$, $\pi_k(K) = 0$ for $0 \le k < n$ and for $k = n + 1$, and $\pi_n(K) = G$. Let Y be a metrizable compactum such that $\dim Y \le n + 1$ and Y is G-acyclic. Then Y is K-acyclic.* $\qquad \square$

5. Extensions of complexes

For the remainder of the paper we shall treat all CW-complexes as being PL and our maps between two of them will be PL. If $e : A \to B$ is a map of CW-complexes, then by $\mathrm{MCY}(e)$ we mean the mapping cylinder of e, where A is identified as a natural subspace in the usual way. When we say "the" generator of a group G, we simply mean that the group can be generated by one element and that we have selected a favored generator in advance. Whenever a CW-complex is connected we are going to suppress the notation of a base-point when designating its homotopy groups.

Let us recall that a Moore space P of type $(\mathbb{Z}/q, n)$, $n \in \mathbb{N}$, $q \in \mathbb{N}_{\ge 2}$, can be described as the quotient space of B^{n+1} under a quotient map which is an embedding of int B^{n+1} and which maps ∂B^{n+1} to S^n as a map of degree q. Such Moore spaces are treated as CW-complexes and we identify S^n as a subspace, denoting it S^n_P. It is well known that

$$\pi_k(P) \approx \begin{cases} 0 & \text{if } k < n, \\ \mathbb{Z}/q & \text{if } k = n \end{cases}$$

and that the inclusion $S^n_P \hookrightarrow P$ induces the quotient homomorphism $\mathbb{Z} \to \mathbb{Z}/q$ on π_n. There is a triangulation L of P so that for some at least three times iterated barycentric subdivision U of B^{n+1} (which we treat as an $(n + 1)$-simplex), the quotient map f of B^{n+1} to P is simplicial from U to L, and so that the L-simplicial neighborhood of S^n_P in P is a regular neighborhood of S^n_P in P. We shall refer to such L as a **standard** triangulation of P.

5.1. Notation. Let $n \in \mathbb{N}$ and K be a simplicial complex. Then $K^{[n+1]}$ will denote $K^{(n+1)} \backslash K^{(n)}$.

5.2. Definition. Let $n \in \mathbb{N}$, K be a simplicial complex and $S = K^{(n+1)}$. For each $\sigma \in S^{[n+1]}$, let $e_\sigma : \partial\sigma \to \Theta_\sigma$ be an embedding into a connected CW-complex Θ_σ. Form a CW-complex M by gluing each Θ_σ to $|S^{(n)}|$ using the map, e_σ. We require, of course, that if $\sigma_1 \neq \sigma_2$, then $\Theta_{\sigma_1} \cap \Theta_{\sigma_2} = \sigma_1 \cap \sigma_2$.

Choose a map $\phi : M \to |S|$ so that

(a) $\phi = \mathrm{id}$ on $|S^{(n)}|$, and

(b) for each $\sigma \in S^{[n+1]}$, $\phi^{-1}(\sigma) = \Theta_\sigma$.

Let us call the resulting pair (M, ϕ) an **extension of $K^{(n)}$ associated with** $\{e_\sigma \mid \sigma \in K^{[n+1]}\}$. Notice that if desired, we may add the condition

(c) $\phi^{-1}(|S^{(n)}|) = |S^{(n)}|$.

The next lemma was influenced by work in [KY2].

5.3. Lemma. *Let $n \in \mathbb{N}_{\geq 2}$, let $q \in \mathbb{N}_{\geq 2}$, let P be a Moore space of type $(\mathbb{Z}/q, n)$, let G be an abelian group, and let $u \in G$ be an element which is divisible by q. Assume that L is a standard triangulation of P, Ψ is a connected CW-complex such that*

$$\pi_k(\Psi) \approx \begin{cases} 0 & \text{if } k < n, \\ G & \text{if } k = n, \end{cases}$$

there is a copy S_Ψ^n of S^n lying in Ψ, and we have selected an embedding $e_\tau : \partial\tau \to S_{\Psi_\tau}^n = S_\Psi^n \subset \Psi_\tau = \Psi$ for each $\tau \in L^{[n+1]}$ such that the induced map $H_n(e_\tau) : H_n(\partial\tau; \mathbb{Z}) \to H_n(\Psi_\tau, \mathbb{Z})$ carries the generator of $H_n(\partial\tau; \mathbb{Z})$ to u (here we identify $H_n(\Psi, \mathbb{Z})$ with $\pi_n(\Psi)$). Let (M, ϕ) be an extension of $L^{(n)}$ associated with $\{e_\tau \mid \tau \in L^{[n+1]}\}$. Then

$$\pi_k(M) \approx \begin{cases} 0 & \text{if } k < n, \\ \mathbb{Z}/q \oplus (\bigoplus_1^r G) & \text{if } k = n, \end{cases}$$

where r is the cardinality of $L^{[n+1]}$.

Proof. It is not difficult to see that M is simply connected. Using a Mayer-Vietoris sequence argument, one can then check that $H_k(M; \mathbb{Z})$ is trivial for all $k < n$. (Begin with S_P^n. Then add the Ψ_τ's back one by one, so that each new one intersects the union of the preceding sets in a subpolyhedron with homology group $H_k = 0$.)

This along with the Hurewicz isomorphism theorem show that it is sufficient to demonstrate that $H_n(M; \mathbb{Z}) \approx \mathbb{Z}/q \oplus (\bigoplus_1^r G)$. Let C be the simplicial neighborhood of S_P^n in P under L and let D be the simplicial neighborhood of ∂B^{n+1} in B^{n+1} under U (notation from the beginning of this section). From the above, each is a regular neighborhood.

Choose $\delta_0 \in U^{[n+1]}$ so that $\partial\delta_0 \cap D = \emptyset$. Then the closure D' of the region between δ_0 and D is an annulus which collapses (simplicially) to its intersection with D. One sees that $f(D' \cup D)$ collapses to C. In turn, C collapses to S_P^n. If we put $\delta_0' = f(\delta_0)$, then we see by this that the complement of the interior of δ_0' in P collapses to S_P^n.

To help understand an argument to follow, suppose that $\delta_0, \ldots, \delta_{r-1}$ is a list of the elements of $U^{[n+1]}$. One can see that the n-simplexes of U can be oriented in such a manner that the n-cycles $\partial\delta_i$ and ∂B^{n+1} satisfy the property that $\partial\delta_0$ is homologous (rel $\mathbb{Z}$) to $\partial\delta_1 + \cdots + \partial\delta_{r-1} + \partial B^{n+1}$. Thus $H_n(|U^{(n)}|; \mathbb{Z})$ is the

sum of r copies of $\mathbb{Z}$ and the inclusion of $\partial \delta_0$ induces a homomorphism on n-th homology which sends the generator of $\partial \delta_0$ to the r-tuple each of whose entries is the generator of $\mathbb{Z}$.

Now the quotient map f is nondegenerate on elements of $U^{[n+1]}$ since it is a homeomorphism on int B^{n+1}. So it is clear that $f(|U^{(n)}|) = |L^{(n)}|$ and that f carries ∂B^{n+1} onto $S_P^n \subset P$ by a map of degree q. If we label $f(\delta_i)$ as δ_i', then we can see by applying the preceding reasoning that $H_n(|L^{(n)}|; \mathbb{Z})$ is isomorphic to the sum of r copies of $\mathbb{Z}$ and that the inclusion j of $\partial \delta_0'$ induces a homomorphism on n-th homology which sends the generator of $H_n(\partial \delta_0'; \mathbb{Z})$ to the r-tuple where say the first coordinate is q and each other is just 1.

Using this information along with an inductive argument involving Mayer-Vietoris sequences, where one considers reversing the collapse of $P \setminus \text{int } \delta_0'$ to S_P^n, we conclude that M may be written as the union of two closed subspaces, say W_1 and W_2, such that

(1) $H_n(W_1; \mathbb{Z}) \approx \mathbb{Z} \oplus (\bigoplus_1^{r-1} G)$,

(2) $H_n(W_2; \mathbb{Z}) \approx G$, and

(3) $W_1 \cap W_2 = \partial \delta_0'$.

Consider the part

$$H_n(\partial \delta_0') \overset{\Delta}{\to} H_n(W_1; \mathbb{Z}) \oplus H_n(W_2; \mathbb{Z}) \to H_n(M) \to 0$$

of the Mayer-Vietoris sequence of the pair (W_1, W_2). This becomes

$$\mathbb{Z} \overset{\Delta}{\to} \mathbb{Z} \oplus (\bigoplus_1^r G) \to H_n(M) \to 0,$$

where Δ is induced by the inclusions $i : \partial \delta_0' \to W_2$, $j : \partial \delta_0' \to W_1$. Clearly $i_*(1) = u$. The discussion preceding this leads us to conclude that $j_*(1) = (q, u, \dots, u) \in \mathbb{Z} \oplus (\bigoplus_1^r G)$. In the Mayer-Vietoris sequence one employs $-j_*$. So our Δ is the same as the one in Lemma 4.1. $\square$

6. Layered extensions and reductions

In this section we are going to describe a method of construction of an extension built as a finite sequence of extensions by the method of section 4. Beginning with a finite simplicial complex C_0, there will first be a finite sequence of extensions starting with $C_0^{(n)}$, each extension being of the same type involving Moore spaces. Then there will be one more extension which might be construed as a trivial Edwards-Walsh resolution. After doing this we shall look at the construction in reverse, determining recursively some of the homotopy groups of special subcomplexes of it, attaching cells to kill off specific summands of these groups, determining homotopy groups again of certain subcomplexes, attaching cells again, and so forth until we construct an object with desirable characteristics.

For the remainder of this section, assume that $n \in \mathbb{N}_{\geq 2}$ is fixed.

Let C_0 be a finite simplicial complex with $\dim C_0 \leq n + 1$. Write $M_0 = |C_0|$. Given $q_1 \in \mathbb{N}_{\geq 2}$, fix a Moore space Θ_1 of type $(\mathbb{Z}/q_1, n)$ along with a collection $\{e_\sigma : \partial \sigma \to S_{\Theta_1}^n \subset \Theta_1 \mid \sigma \in C_0^{[n+1]}\}$ of embeddings. Form the extension (M_1, φ_1) of $C_0^{(n)}$ associated with $\{e_\sigma \mid \sigma \in C_0^{[n+1]}\}$.

Suppose we are given $q_2 \in \mathbb{N}_{\geq 2}$ and a triangulation C_1 of M_1 in such a manner that $C_1|\varphi_1^{-1}(\sigma)$ is standard for each $\sigma \in C_0^{[n+1]}$. Also require that if σ, $\sigma_1 \in C_0^{[n+1]}$, then there is a simplicial isomorphism between $C_1|\varphi_1^{-1}(\sigma)$ and $C_1|\varphi_1^{-1}(\sigma_1)$. Let Θ_2 be a Moore space of type $(\mathbb{Z}/q_2, n)$ along with a collection $\{e_\sigma : \partial\sigma \to S_{\Theta_2}^n \subset \Theta_2 \,|\, \sigma \in C_1^{[n+1]}\}$ of embeddings. Form the extension (M_2, φ_2) of $C_1^{(n)}$ associated with $\{e_\sigma \,|\, \sigma \in C_1^{[n+1]}\}$.

Such a procedure may be applied recursively, say m times, finally resulting in an extension (M_m, φ_m) of $C_{m-1}^{(n)}$ associated with $\{e_\sigma \,|\, \sigma \in C_{m-1}^{[n+1]}\}$ along with a triangulation C_m of M_m such that $C_m|\varphi_m^{-1}(\sigma)$ is standard for each $\sigma \in C_{m-1}^{[n+1]}$.

6.1. Definition. We shall refer to such a sequence

$$M(C_0) = \big((M_1, \varphi_1), \ldots, (M_m, \varphi_m)\big)$$

as an m-**layer** $\{q_1, \ldots, q_m\}$-**extension** of $C_0^{(n)}$. Call $\{C_1, \ldots, C_m\}$ its **triangulations**.

6.2. Lemma. *Suppose $M(C_0)$ is an m-layer $\{q_1, \ldots, q_m\}$-extension of $C_0^{(n)}$ with notation as in Definition 6.1. Then for each $1 \leq j \leq m$,*

    (a) $\dim M_j \leq n + 1$,
    (b) $M_{j-1}^{(n)} \subset M_j^{(n)}$,
    (c) $M_j^{(n)} = |C_{j-1}^{(n)}|$,
    (d) *if $\sigma \in C_{j-1}^{[n+1]}$, then $\partial\sigma \subset M_j^{(n)}$, and*
    (e) M_j *is compact and is connected if C_0 is connected.*

Suppose, moreover, that V_0 is a subcomplex of C_0. Put $N_0 = |V_0|$ and for each $1 \leq i \leq m$, $N_i = \varphi_i^{-1}(N_{i-1})$, $V_i = C_i|N_i$, and $\psi_i = \varphi_i|N_i : N_i \to N_{i-1}$. Then,

    (f) $\big((N_1, \psi_1), \ldots, (N_m, \psi_m)\big)$ *is an m-layer $\{q_1, \ldots, q_m\}$-extension of V_0 with triangulations $\{V_1, \ldots, V_m\}$.*

6.3. Lemma. *Let C_0 be a simplicial complex consisting of an $(n+1)$-simplex Δ along with all its faces, and let $M(C_0)$ be an m-layer $\{q_1, \ldots, q_m\}$-extension of $C_0^{(n)}$ with notation as in Definition 6.1. For each $\sigma \in \bigcup_{j=1}^{m} C_{j-1}^{[n+1]}$, let $j_\sigma : \partial\sigma \to M_m^{(n)}$ denote the inclusion map and $\alpha_\sigma \in \pi_n(M_m^{(n)})$ the image of the generator of $\pi_n(\partial\sigma)$ under the induced homomorphism $\pi_n(j_\sigma)$. Then, $\{\alpha_\sigma \,|\, \sigma \in \bigcup_{j=1}^{m} C_{j-1}^{[n+1]}\}$ is a generating set for $\pi_n(M_m^{(n)})$, $\pi_n(M_m)$ is a torsion group having nontrivial q_m-torsion, and $\pi_k(M_m^{(n)}) = 0$ for $0 \leq k < n$.*

Proof. Suppose $m = 1$. From Lemma 6.2(c), the fact that Δ is an $(n+1)$-simplex, the attached object is a Moore space, and since $n \geq 2$, one sees that the lemma is true.

Consider the case $m = 2$. Lemma 6.2(e) shows that $\pi_0(M_m^{(n)}) = 0$. We leave to the reader the easily proved fact that $\pi_1(M_m^{(n)}) = 0$. For the rest of this proof all homology will be with respect to $\mathbb{Z}$. For $\sigma \in \bigcup_{j=1}^{m} C_{j-1}^{[n+1]}$, let $\hat{\alpha}_\sigma \in H_n(M_m^{(n)})$ designate the image of the generator of $H_n(\partial\sigma)$ in $H_n(M_m^{(n)})$ under the induced homomorphism $H_n(j_\sigma)$. We shall show that $H_k(M_m^{(n)}) = 0$ for $2 \leq k < n$ and

$\{\hat{\alpha}_\sigma \,|\, \sigma \in \bigcup\limits_{j=1}^{m} C_{j-1}^{[n+1]}\}$ generates $H_n(M_m^{(n)})$. Then an application of the Hurewicz Theorem ([Hu]) would complete the proof about the generating set for $\pi_n(M_m^{(n)})$ for this case.

Clearly M_1 consists of $M_1^{(n)}$ along with some Moore spaces, say $E_1, \ldots, E_r$ glued respectively along $\partial\sigma_1, \ldots, \partial\sigma_r$, where $\sigma_1, \ldots, \sigma_r$ varies in $C_0^{[n+1]}$. Each E_s has a triangulation F_s induced by C_1. Since E_s is a Moore space, then $H_k(|F_s^{(n)}|) = 0$, $2 \leq k < n$. Let $\tilde{j}_s : \partial\sigma_s \to |F_s^{(n)}|$, $1 \leq s \leq r$, and $\tilde{j}_\tau : \partial\tau \to |F_s^{(n)}|$, $\tau \in F_s^{[n+1]}$, denote the inclusion maps. Put $\tilde{\alpha}_s$ equal to the image in $H_n(|F_s^{(n)}|)$ of the generator of $H_n(\partial\alpha_s)$ under $H_n(\tilde{j}_s)$ and for each $\tau \in F_s^{[n+1]}$, put $\tilde{\alpha}_\tau$ equal to the image in $H_n(|F_s^{(n)}|)$ of the generator of $H_n(\partial\tau)$ under $H_n(\tilde{j}_\tau)$. Then $\{\tilde{\alpha}_s\} \cup \{\tilde{\alpha}_\tau \,|\, \tau \in F_s^{[n+1]}\}$ is a generating set for $H_n(|F_s^{(n)}|)$.

Now use induction, beginning with $M_1^{(n)}$ and successively gluing in $|F_s^{(n)}|$, $1 \leq s \leq r$. At each stage use a Mayer-Vietoris sequence to deduce that $H_k = 0$ for $2 \leq k < n$, and also to detect a generating set for H_n. At the last instance, one will have found the desired generating set for $\pi_n(M_2^{(n)})$ and that $\pi_k(M_2^{(n)}) = 0$ for $0 \leq k < n$.

The torsion part follows from the fact that in $H_n(M_m^{(n)})$, $\hat{\alpha}_\Delta$ is a multiple of the sum of the other generators, $\hat{\alpha}_\sigma$. Of course each of the latter is transformed to an element of torsion in $\pi_n(M_m)$ since, geometrically, it includes as the generator of π_n of an embedded Moore space.

The preceding can be applied recursively; therefore the lemma is true for arbitrary $m \in \mathbb{N}$. $\qquad\square$

For the rest of this section, assume we have been given a C_0 and $M(C_0)$ as in Definition 6.1. Also, let G be an abelian group, $u \in G$, and suppose that u is divisible by q_i, $1 \leq i \leq m$. We are going to design, in association with $M(C_0)$ and its specific structure, a sequence $(D_0, \omega_0), \ldots, (D_m, \omega_m)$ of CW-complexes and maps. This will be arranged so as to decrease the complexity of the homotopy groups of certain subcomplexes of these D_k's, but in a very systematic manner. We shall apply this technology in section 7 below.

Fix a CW-complex Φ such that $\Phi^{(n-1)}$ is a vertex and

$$\pi_k(\Phi) \approx \begin{cases} 0 & \text{if } k < n, \\ G & \text{if } k = n. \end{cases}$$

For each $\sigma \in C_m^{[n+1]}$, let $d_\sigma : \partial\sigma \to \Phi$ be a map such that $\pi_n(d_\sigma)$ sends the generator of $\pi_n(\partial\sigma)$ to u. Denote by Φ_σ the mapping cylinder, $\mathrm{MCY}(d_\sigma)$, and by e_σ the inclusion of $\partial\sigma$ into Φ_σ. Select an extension (D_0, ω_0) of $C_m^{(n)}$ associated with $\{e_\sigma \,|\, \sigma \in C_m^{[n+1]}\}$.

Recall that M_m is endowed with the triangulation C_m. We may choose a triangulation T of D_0 and adjust the map ω_0 so that

(a$_1$) for each $\sigma \in C_m^{[n+1]}$, T triangulates $\omega_0^{-1}(\sigma)$, and
(b$_1$) $\omega_0 : |T| \to |C_m|$ is simplicial.

Consider now any $\sigma \in C_{m-1}^{[n+1]}$. An application of Lemma 5.3 shows that

$$\pi_k(\omega_0^{-1}(\varphi_m^{-1}(\sigma))) \approx \begin{cases} 0 & \text{if } k < n, \\ \mathbb{Z}/q_m \oplus (\overset{r_1}{\underset{1}{\bigoplus}}G) & \text{if } k = n, \end{cases}$$

for some $r_1 \in \mathbb{N}$, where r_1 is independent of σ.

Attach an $(n+1)$-cell to $\omega_0^{-1}(\varphi_m^{-1}(\sigma)) \subset D_0$ via a map on ∂B^{n+1} which induces a homomorphism sending the generator of $\pi_n(\partial B^{n+1})$ to the generator of the summand $\mathbb{Z}/q_m$ of $\pi_n(\omega_0^{-1}(\varphi_m^{-1}(\sigma)))$. Let us designate such an $(n+1)$-cell by B_σ. We shall use D_1 to denote the CW-complex obtained from D_0 after attaching all the $(n+1)$-cells B_σ for $\sigma \in C_{m-1}^{[n+1]}$.

There is a map $\omega_1 : D_1 \to M_{m-1} = |C_{m-1}|$ which equals $\varphi_m \circ \omega_0$ on D_0 and which sends B_σ into σ for each $\sigma \in C_{m-1}^{[n+1]}$. The first thing to note is that

$$\pi_k(\omega_1^{-1}(\sigma)) \approx \begin{cases} 0 & \text{if } k < n, \\ \overset{r_1}{\underset{1}{\bigoplus}}G & \text{if } k = n, \end{cases}$$

for each $\sigma \in C_{m-1}^{[n+1]}$. Indeed, the inclusion induced homomorphism of $\pi_n(\partial\sigma)$ into $\pi_n(\omega_1^{-1}(\sigma))$ sends the generator of the former to an element u^* of $\overset{r_1}{\underset{1}{\bigoplus}}G$ which is independent of σ and so that u^* is divisible by q_i, $1 \le i \le m$.

The reader should observe that for such σ, $\omega_1^{-1}(\sigma) = \omega_0^{-1}(\varphi_m^{-1}(\sigma)) \cup B_\sigma$. Moreover, suppose that $f : B^{n+1} \to B_\sigma$ is a map having the property that $f(\partial B^{n+1}) \subset \omega_0^{-1}(\varphi_m^{-1}(\sigma))$. Let α be the generator of $\pi_n(\partial B^{n+1})$ and $g = f|\partial B^{n+1} : \partial B^{n+1} \to \omega_0^{-1}(\varphi_m^{-1}(\sigma))$.

In the quotient map $\mathbb{Z} \to \mathbb{Z}/q_m$, note that $q_m = q_m \cdot 1$ is mapped to 0. This shows that

(†) $\pi_n(g)(\alpha)$ is divisible by q_m in $\pi_n(\omega_0^{-1}(\varphi_m^{-1}(\sigma)))$.

If g is a map of ∂B^{n+1} to $\omega_0^{-1}(\varphi_m^{-1}(\sigma))$ and if $\pi_n(g)$ sends the generator of $\pi_n(\partial B^{n+1})$ to $q_m \cdot r$ times the generator of $\mathbb{Z}/q_m$, then it sends that generator to zero, so g is nullhomotopic. Hence

(††) if g factors through a map of ∂B^{n+1} to S^n of degree which is a multiple of q_m, then $f|\partial B^{n+1}$ extends to a map $f_0 : B^{n+1} \to \omega_0^{-1}(\varphi_m^{-1}(\sigma))$.

Now consider $\sigma \in C_{m-2}^{[n+1]}$. This time we apply Lemma 5.3 with the group $G' = \overset{r_1}{\underset{1}{\bigoplus}}G$ and the element $u^* \in G$ which is divisible by q_{m-1}. One concludes that

$$\pi_k(\omega_1^{-1}(\varphi_{m-1}^{-1}(\sigma))) \approx \begin{cases} 0 & \text{if } k < n, \\ \mathbb{Z}/q_{m-1} \oplus (\overset{r_1 r_2}{\underset{1}{\bigoplus}}G) & \text{if } k = n, \end{cases}$$

for some $r_2 \in \mathbb{N}$ which is independent of σ.

Then of course we will form a CW-complex D_2 by "killing off" the $\mathbb{Z}/q_{m-1}$ summands as in the previous construction by attaching cells denoted B_σ, $\sigma \in C_{m-2}^{[n+1]}$ respectively to $\omega_1^{-1}(\varphi_{m-1}^{-1}(\sigma))$, and end with a map $\omega_2 : D_2 \to M_{m-2} = |C_{m-2}|$

similar to ω_1 above. One should observe by analogy that

$$\pi_k(\omega_2^{-1}(\sigma)) \approx \begin{cases} 0 & \text{if } k < n, \\ \bigoplus_1^{r_1 r_2} G & \text{if } k = n. \end{cases}$$

Also, $\omega_2^{-1}(\sigma) = \omega_1^{-1}(\varphi_{m-1}^{-1}(\sigma)) \cup B_\sigma$, and statements similar to (†) and (††) will be true.

This process obviously can be continued recursively until we have a sequence of maps, $\omega_j : D_j \to |C_{m-j}|$, $0 \le j \le m$. In particular several conditions will be true of this construction. Suppose that $1 \le j \le m$; then

(1) $|C_0^{(n)}| \subset D_m$, $D_{j-1} \subset D_j$ and $D_j \backslash D_{j-1} = \bigcup\{\text{int } B_\sigma \mid \sigma \in C_{m-j}^{[n+1]}\}$,

(2) $\omega_j | D_{j-1} = \varphi_{m-j+1} \circ \omega_{j-1}$,

(3) for each $\sigma \in C_{m-j}^{[n+1]}$, $\omega_j^{-1}(\sigma) = \omega_{j-1}^{-1}(\varphi_{m-j+1}^{-1}(\sigma)) \cup B_\sigma$, and

(4) for each $\sigma \in C_0^{[n+1]}$, there exists $r \in \mathbb{N}$ (which may depend on σ) such that

$$\pi_k(\omega_m^{-1}(\sigma)) \approx \begin{cases} 0 & \text{if } k < n, \\ \bigoplus_1^r G & \text{if } k = n. \end{cases}$$

Moreover, suppose that $\sigma \in C_{m-j}^{[n+1]}$ and $f : B^{n+1} \to B_\sigma$ is a map having the property that $f(\partial B^{n+1}) \subset \omega_{j-1}^{-1}(\varphi_{m-j+1}^{-1}(\sigma))$. Then

(5) $f | \partial B^{n+1} : \partial B^{n+1} \to \omega_{j-1}^{-1}(\varphi_{m-j+1}^{-1}(\sigma))$ is of degree a multiple of q_{m-j+1}, and

(6) if $f | \partial B^{n+1}$ factors through a map of ∂B^{n+1} to S^n of degree which is a multiple of q_{m-j+1}, then $f | \partial B^{n+1}$ extends to a map $f_0 : B^{n+1} \to \omega_{j-1}^{-1}(\varphi_{m-j+1}^{-1}(\sigma))$.

6.4. Definition. We shall refer to a sequence $\big((D_0, \omega_0), \ldots, (D_m, \omega_m)\big)$ of the type just constructed, satisfying (1)–(6), as a **reduction** of $M(C_0)$ for (the abelian group) G.

6.5. Lemma. *Assume that $\{q_1, \ldots, q_m\} \subset \mathbb{N}_{\ge 2}$, G is an abelian group, $u \in G$, and u is divisible by q_i for each i. Then for each simplicial complex C_0 with $\dim C_0 \le n + 1$ and m-layer $\{q_1, \ldots, q_m\}$-extension $M(C_0)$ with triangulations $C_1, \ldots, C_m$, there exists a reduction $\big((D_0, \omega_0), \ldots, (D_m, \omega_m)\big)$ of $M(C_0)$ for G. For each $\sigma \in C_m^{[n+1]}$, $\omega_0^{-1}(\sigma) = \mathrm{MCY}(d_\sigma)$, where $d_\sigma : \partial\sigma \to \Phi$, and $\pi_n(d_\sigma)$ carries the generator of $\pi_n(\partial\sigma)$ to $u \in G \approx \pi_n(\Phi)$. The CW-complex Φ is chosen so that*

$$\pi_k(\Phi) \approx \begin{cases} 0 & \text{if } k < n, \\ G & \text{if } k = n. \end{cases} \qquad \square$$

6.6. Lemma. *Assume that $q \in \mathbb{N}_{\ge 2}$ and $G = \mathbb{Z}/q$. Then for each simplicial complex C_0 with $\dim C_0 \le n+1$ and m-layer $\{q_1, \ldots, q_m\}$-extension $M(C_0)$, there exists a reduction $\big((D_0, \omega_0), \ldots, (D_m, \omega_m)\big)$ of $M(C_0)$ for G.*

Proof. We only need to make a few adjustments to the construction preceding Definition 6.4. In producing D_0, this time let Φ be a Moore space of type of $(\mathbb{Z}/q, n)$, d_σ sending generator to generator. Then, up to a homotopy equivalence, $\big((M_1, \varphi_1), \ldots, (M_m, \varphi_m), (D_0, \omega_0)\big)$ is an $(m + 1)$-layer $\{q_1, \ldots, q_m, q\}$-extension of $C_0^{(n)}$. Applying Lemmas 6.2(f) and 6.3, for each $\sigma \in C_{m-1}^{[n+1]}$, $\pi_n(\omega_0^{-1}(\varphi_m^{-1}(\sigma)))$ is

a finitely generated torsion group with nontrivial q-torsion. This time attach an $(n+1)$-cell B_σ to kill all but the q-torsion. This process can be continued recursively until the desired conclusion is reached. $\qquad\square$

7. Main construction step

Our proof of Theorem 1.1 in section 8 will involve an inductive construction. There is one main, quite technical step in that procedure, which we would like to present in isolation now.

For the remainder of this section, assume that $n \in \mathbb{N}$, $\mathbf{Y} = (|L_i|, p_i^{i+1})$ is an inverse sequence of finite polyhedra, $Y = \lim \mathbf{Y}$, and for each $i \in \mathbb{N}$,

(i) $\dim L_i \leq n+1$, and
(ii) $p_i^{i+1} : |L_{i+1}| \to |L_i|$ is a simplicial surjection.

Put $C_0 = L_1$, and suppose that $M(C_0) = \big((M_1, \varphi_1), \ldots, (M_m, \varphi_m)\big)$ is an m-layer $\{q_1, \ldots, q_m\}$-extension of $C_0^{(n)}$ (see Definition 6.1), with triangulations C_i, $1 \leq i \leq m$. Assume that we are given a subcomplex N of C_m with

(iii) $C_m^{(n)} \subset N$.

7.1. Lemma. *Let $P_0 \subset \mathbb{P}$ and $\{q_0, q_1, \ldots, q_m\} \subset \mathbb{N}$ be selected so that each q_i is a composite of elements of $\mathbb{P}\backslash P_0$, and either $G = \mathbb{Z}_{(P_0)}$ or for some $p \in \mathbb{P}\backslash P_0$, $G \in \{\mathbb{Z}/p, \mathbb{Z}/p^\infty\}$. Assume that $\dim_G Y \leq n$. Then there exists $s > 1$, such that for all $\epsilon > 0$ there is a subdivision V_0 of L_s, an $(m+1)$-layer $\{q_1', \ldots, q_{m+1}'\}$-extension $M(V_0^{(n+1)}) = \big((N_1, \psi_1), \ldots, (N_{m+1}, \psi_{m+1})\big)$ of $V_0^{(n)}$ with triangulations V_i, $1 \leq i \leq m+1$, and a map $g : N_{m+1} \to M_m$ such that:*

(a) *$\operatorname{mesh} V_0 < \epsilon$,*
(b) *q_i' is a composite of elements of $\mathbb{P}\backslash P_0$, and q_0 divides q_i' for each $1 \leq i \leq m+1$,*
(c) *for each $\sigma_0 \in \bigcup\limits_{j=1}^{m} V_{j-1}^{[n+1]}$, $g|\partial\sigma_0 : \partial\sigma_0 \to M_m$ is of degree a multiple of q_0,*
(d) *if $G \neq \mathbb{Z}_{(P_0)}$ and $\sigma' \in V_m$, then either $\psi_{m+1}^{-1}(\sigma') \subset g^{-1}(|N|)$ or $\sigma' \in V_m^{[n+1]}$ and $\psi_{m+1}^{-1}(\sigma') \cap g^{-1}(|N|) = \partial\sigma'$, and*
(f) *in case $G = \mathbb{Z}_{(P_0)}$ and $h : |N| \to K(G, n)$ is a map, then $h \circ g|g^{-1}(|N|) : g^{-1}(|N|) \to K(G, n)$ extends to a map of N_{m+1} to $K(G, n)$.*

Moreover, let $\varphi = \varphi_1 \circ \cdots \circ \varphi_m : M_m \to |L_1|$ and $\psi = \psi_1 \circ \cdots \circ \psi_{m+1} : N_{m+1} \to |L_s|$:

(g) *then for each $z \in N_{m+1}$, there is a principal simplex $\sigma^{\#}$ of L_1 such that if $p_1^s \circ \psi(z)$ lies in $\sigma^{\#}$, then $\varphi \circ g(z) \in \sigma^{\#}$.*

Proof. Let us use $p_k : Y \to |L_k|$ for the coordinate projections. Observe that because of (ii),

(1) for each $k \geq 1$, the coordinate projection p_k is surjective.

If $G \in \{\mathbb{Z}/p^\infty, \mathbb{Z}_{(P_0)}\}$, then we shall pick an element $u \in G$ as follows. Should $G = \mathbb{Z}/p^\infty$, choose $u = 0 \in G$. Otherwise select $\alpha : \mathbb{Z} \to \mathbb{Z}_{(P_0)}$ as in Lemma 4.5, and choose $u = \alpha(1)$. In either case (see Lemma 4.4), the hypotheses of Lemma 6.5 apply. If $G = \mathbb{Z}/p$, then Lemma 6.6 comes into play. Hence we may select a reduction $\big((D_0, \omega_0), (D_1, \omega_1), \ldots, (D_m, \omega_m)\big)$ of $M(C_0)$ for G. Let us agree that if $G = \mathbb{Z}_{(P_0)}$, then in forming (D_0, ω_0) we shall use a CW-complex Φ as in Lemma 6.5, and so that $u = \alpha(1) \in G \approx \pi_n(\Phi)$ is as above.

As a consequence of (i), $\dim Y \leq n+1$, and we are given that $\dim_G Y \leq n$. One then may apply (4) in Definition 6.4 along with Lemma 2.5 to see that $\operatorname{extdim} Y \leq \omega_m^{-1}(\sigma)$ for each $\sigma \in C_0^{[n+1]}$. Putting it another way, we may say that

(2) for each $\sigma \in C_0^{[n+1]}$, $\omega_m^{-1}(\sigma)$ is an absolute extensor for Y.

We claim that there exists $s > 1$ and a map $\beta_0 : |L_s| \to D_m$ such that

(3) if $y \in Y$ and $p_1(y) \in \sigma$, where σ is a principal simplex of L_1, then $\beta_0 \circ p_s(y) \in \omega_m^{-1}(\sigma)$.

We shall first get a map $\gamma : Y \to D_m$. Applying (1) of Definition 6.4, define γ on $p_1^{-1}(C_0^{(n)})$ to equal p_1. Now if $\sigma \in C_0^{[n+1]}$, consider $W_\sigma = \omega_m^{-1}(\sigma)$. Then $\gamma(p_1^{-1}(\partial \sigma)) = p_1(p_1^{-1}(\partial \sigma)) \subset \partial \sigma \subset W_\sigma$. Using (2), extend γ to a map of $p_1^{-1}(\sigma)$ into W_σ. Put all these extensions together to get $\gamma : Y \to D_m$ so that

(4) if $y \in Y$ and $p_1(y) \in \sigma$, where σ is a principal simplex of C_0, then $\gamma(y) \in \omega_m^{-1}(\sigma)$.

There exists $s > 1$ and a map $\beta_0 : |L_s| \to D_m$ such that the composition $\beta_0 \circ p_s$ is as close to γ as we wish. For each principal simplex σ of C_0, $\omega^{-1}(\sigma)$ is a subcomplex of D_m. So in light of (4) modify the map β_0 if necessary so that (3) is true.

As a consequence of (3) and (1), we may conclude that

(5) if $t \in |L_s|$ and $p_1^s(t) \in \sigma$, where σ is a principal simplex of L_1, then $\beta_0(t) \in \omega_m^{-1}(\sigma)$.

Next we shall begin the construction of the objects (N_k, ψ_k), $1 \leq k \leq m+1$. The first m steps of this program are designed to "peel off" the cells B_σ from the target space in a certain gradual manner. We will start by replacing $\beta_0 : |L_s| \to D_m$ by a map $\beta_1 : N_1 \to D_{m-1}$ and then proceed recursively so that in the end, no B_σ will lie in the image of the last map $\beta_m : N_m \to D_0$.

Now we apply (1) of Definition 6.4. Taking into account (5) above, one may choose a subdivision V_0 of L_s satisfying (a), perhaps adjust the map β_0 slightly, and conclude that

(6) $\beta_0(|V_0^{(n)}|) \subset D_0$,

(7) if $\sigma_0 \in V_0$, then for some principal simplex $\sigma_0^\# \in C_0$, $\beta_0(\sigma_0) \subset \omega_m^{-1}(\sigma_0^\#)$, and

(8) if $\sigma_0 \in V_0^{[n+1]}$, then either $\beta_0(\sigma_0) \subset D_{m-1}$ or $\beta_0(\sigma_0) = B_\sigma$ for some $\sigma \in C_0^{[n+1]}$.

We claim that

(9) for each $\sigma_0 \in V_0^{[n+1]}$, $\beta_0|\partial \sigma_0 : \partial \sigma_0 \to D_{m-1}$ is of degree a multiple of q_m.

According to (8), there are but two possibilities. In case $\beta_0(\sigma_0) \subset D_{m-1}$, then (9) is true because $\beta_0|\partial \sigma_0 : \partial \sigma_0 \to D_{m-1}$ is nullhomotopic. If $\beta_0(\sigma_0) = B_\sigma$ for some $\sigma \in C_0^{[n+1]}$, then an application of (5) of Definition 6.4 along with (6) of this section yields (9).

Choose
$$q_1' = q_0 q_m.$$

Put $N_0 = |V_0|$. Fix a Moore space Φ_1 of type $(\mathbb{Z}/q_1', n)$ along with a collection $\{e_\sigma : \partial \sigma \to S_{\Phi_1}^n \subset \Phi_1 \,|\, \sigma \in V_0^{[n+1]}\}$ of embeddings. Form the extension (N_1, ψ_1) associated with $\{e_\sigma \,|\, \sigma \in V_0^{[n+1]}\}$.

A map $\beta_1 : N_1 \to D_{m-1}$, having the property

(10) if $\sigma_0 \in V_0$, then for some principal simplex $\sigma_0^\# \in C_0$, $\beta_1(\psi_1^{-1}(\sigma_0)) \subset \omega_{m-1}^{-1}(\varphi_1^{-1}(\sigma_0^\#))$,

is described as follows. Consider $\sigma_0 \in V_0$. We will define β_1 in two mutually exclusive, exhaustive cases. First if $\beta_0(\sigma_0) \subset D_{m-1}$, then on $\psi_1^{-1}(\sigma_0)$, put $\beta_1 = \beta_0 \circ \psi_1$. Then (10) holds because of (7) and item (2) of Definition 6.4.

In the other situation, we first define $\beta_1 = \beta_0 \circ \psi_1 = \beta_0$ on $\partial\sigma_0$. We then use (6) of Definition 6.4, with f replaced by β_0, to get an extension of β_1 to the entire Moore space $\psi_1^{-1}(\sigma_0)$, thought of as a quotient of B^{n+1} under a certain map which has degree q_1' on ∂B^{n+1}. One sees from this definition that again (10) is true. Also, (9) obtains if we replace β_0 by β_1, i.e.,

(11) for each $\sigma_0 \in V_0^{[n+1]}$, $\beta_1|\partial\sigma_0 : \partial\sigma_0 \to D_{m-1}$ is of degree a multiple of q_m.

Surely one effect of this construction is to "peel off" the cells B_σ, $\sigma \in C_0^{[n+1]}$, from the image of this new map β_1.

We may select a triangulation V_1 of N_1, slightly (up to homotopy) modify the map $\beta_1 : N_1 \to D_{m-1}$ if necessary, and conclude that along with (10) and (11),

(12) $\beta_1(|V_1^{(n)}|) \subset D_0$, and

(13) if $\sigma_0 \in V_1^{[n+1]}$, then either $\beta_1(\sigma_0) \subset D_{m-2}$ or $\beta_1(\sigma_0) = B_\sigma$ for some $\sigma \in C_1^{[n+1]}$.

Now choose

$$q_2' = q_0 q_{m-1}.$$

Fix a Moore space Φ_2 of type $(\mathbb{Z}/q_2', n)$ along with a collection $\{e_\sigma : \partial\sigma \to S_{\Phi_2}^n \subset \Phi_2 \,|\, \sigma \in V_1^{[n+1]}\}$ of embeddings, and form the extension (N_2, ψ_2) associated with $\{e_\sigma \,|\, \sigma \in V_1^{[n+1]}\}$. There is no loss of generality in assuming that $V_1|\psi_1^{-1}(\sigma)$ is standard for each $\sigma \in V_0^{[n+1]}$.

Replacing β_0 by β_1, apply a procedure just like the one used above to get a map $\beta_2 : N_2 \to D_{m-2}$, agreeing with β_1 on $|V_1^{(n)}|$ and satisfying:

(14) for each $\sigma_0 \in \bigcup_{j=0}^{1} V_j^{[n+1]}$, $\beta_2|\partial\sigma_0 : \partial\sigma_0 \to D_{m-2}$ is of degree a multiple of q_{m-1}, and

(15) if $\sigma_0 \in V_0$, then for some principal simplex $\sigma_0^\# \in C_0$, $\beta_2(\psi_2^{-1}(\psi_1^{-1}(\sigma_0))) \subset \omega_{m-2}^{-1}(\varphi_2^{-1}(\varphi_1^{-1}(\sigma_0^\#)))$.

Such a process may be applied recursively, resulting in extensions $(N_1, \psi_1), \ldots, (N_m, \psi_m)$, triangulations $V_1, \ldots, V_{m-1}$ (we have not yet selected V_m), maps $\beta_1, \ldots, \beta_m$ $(\beta_i : N_i \to D_{m-i})$, and $q_1', \ldots, q_m'$. The following conditions will hold true from this process:

(16) q_0 divides q_i' for each $1 \le i \le m$,

(17) for each $\sigma_0 \in \bigcup_{j=1}^{m} V_{j-1}^{[n+1]}$, $\beta_m|\partial\sigma_0 : \partial\sigma_0 \to D_0$ is of degree a multiple of q_i for some $1 \le i \le m$,

(18) if $\sigma_0 \in V_0$, then for some principal simplex $\sigma_0^\#$ of C_0,

$$\beta_m((\psi_1 \circ \cdots \circ \psi_m)^{-1}(\sigma_0)) \subset \omega_0^{-1}((\varphi_1 \circ \cdots \circ \varphi_m)^{-1}(\sigma_0^\#)) = \omega_0^{-1}(\varphi^{-1}(\sigma_0^\#)),$$

and

(19) $V_i|\psi_i^{-1}(\sigma)$ is standard for each $\sigma \in V_{i-1}^{[n+1]}$ and $1 \le i \le m$.

Recall from section 6 (see (a_1) and (b_1)) that D_0 may be assigned a triangulation T and ω_0 chosen so that $\omega_0 : |T| \to |C_m|$ is simplicial. Thus we may impose a triangulation V_m on N_m so that $V_m|\psi_m^{-1}(\sigma)$ is standard for each $\sigma \in V_{m-1}$, and we may assume (by a simplicial approximation) that, along with (17) and (18), $\beta_m : |V_m| \to |T|$ is simplicial. Hence we may make the above choices of ω_0 and β_m so that

(20) if $G \neq \mathbb{Z}_{(P_0)}$, the map $\omega_0 \circ \beta_m : |V_m| \to |C_m|$ is simplicial.

By applying (c) of Definition 5.2, we require that

(21) if $G = \mathbb{Z}_{(P_0)}$, then $\omega_0^{-1}(|C_m^{(n)}|) = |C_m^{(n)}|$.

We put a triangulation V_m on N_m so that $V_m|\psi_m^{-1}(\sigma)$ is standard for each $\sigma \in V_{m-1}$.

Set $q'_{m+1} = q_0$ and let Φ_{m+1} be a Moore space of type $(\mathbb{Z}/q'_{m+1}, n)$. Choose a collection $\{e_\sigma : \partial\sigma \to S^n_{\Phi_{m+1}} \subset \Phi_{m+1} \,|\, \sigma \in V_m^{[n+1]}\}$ of embeddings, and form the extension (N_{m+1}, ψ_{m+1}) associated with $\{e_\sigma \,|\, \sigma \in V_m^{[n+1]}\}$. We require now, for this extension, the condition (c) of Definition 5.2, i.e.,

(22) $\psi_{m+1}^{-1}(|V_m^{(n)}|) = |V_m^{(n)}|$.

We have already established (a), and certainly our choice of the numbers $q'_1, \ldots,$ q'_{m+1} meets the requirement (b) of this lemma. We define

$$g = \omega_0 \circ \beta_m \circ \psi_{m+1} : N_{m+1} \to M_m.$$

An application of (17) and the fact that β_m is a factor in the composition defining g show that (c) is true. One arrives at (d) because of (20), (22), and since N is a subcomplex of C_m containing $C_m^{(n)}$.

For (f), we examine $h \circ \omega_0|\omega_0^{-1}(|N|) : \omega_0^{-1}(|N|) \to K(G, n)$. Now the CW-complex Φ may be thought of as a subcomplex of $K(G, n)$ since it was chosen as in Lemma 6.5. If $\sigma \in C_m^{[n+1]}$, then (21), our choice of α, and an application of Lemma 4.5 show that if $\sigma \notin N$, then the map $h \circ \omega_0|\partial\sigma$ extends to a map of $\omega_0^{-1}(\sigma) = \Phi$ to $K(G, n)$. Since ω_0 is a factor of g, then (f) follows.

The last statement, (g), can be traced from (18), the definition of ψ_{m+1}, and the factorization of g. □

8. Proof of the main theorem

Proof of Theorem 1.1. Recall from section 4 that $P_G = \{p \in \mathbb{P} \,|\, \mathbb{Z}_{(p)} \in \sigma(G)\}$. If $P_G = \mathbb{P}$, then an application of Example 1 in section 2 (page 8) of [Dr6] shows that $\dim_{\mathbb{Z}} X \leq n$. In that case, we could apply the Edwards-Walsh resolution theorem [Wa] to get a cell-like resolution $\pi : Z \to X$, where $\dim Z \leq n$; of course each fiber of π would be G-acyclic. Hence (a)-(c) of Theorem 1.1 would be satisfied in an even stronger manner. Thus, let us continue under the assumption that $\mathbb{P}\backslash P_G \neq \emptyset$.

Choose a function $\mu : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ which is surjective and each of whose fibers is infinite, always denoting $\mu(i)$ as (u_i, v_i). We do this so that

(1) $u_i \leq i$ for each i.

We may assume that $\operatorname{card} X \geq 2$. Using Lemma 3.7, let $\mathbf{K}_0 = (|K_{(0)i}|, p_{(0)i}^{i+1})$ be a stably irreducible representation of X which is simplicial from index 1. For each $k \in \mathbb{N}$, let

$$\mathcal{M}_k = \overline{\operatorname{st}}K_{(0)k}.$$

We are going to describe a certain recursive process in which there will be an increasing function $r : \mathbb{N} \to \mathbb{N}$, $r(1) = 1$. This will show us where to make a sequence of $r(i)$-shifts beginning with $\mathbf{K}_0$. There will also be an inverse sequence (M_i, g_i^{i+1}), where each M_i is a finite polyhedron with $\dim M_i \leq n + 1$. The limit of this inverse sequence will be the space Z, but in order to produce the requested map $\pi : Z \to X$, it will be necessary to construct the sequence carefully, providing a complicated set of inductive hypotheses along the way.

We shall use the convention that whenever we have determined a space M_i, then $g_i^i : M_i \to M_i$ will denote the identity map. Begin the process by choosing $r(1) = 1$. Let Σ_1 be a subdivision of $K_{(0)1}$ such that $\operatorname{mesh} \Sigma_1 < \lambda^u(\mathcal{M}_1)/4$. Define S_1 to be the restriction of Σ_1 to $|K_{(0),1}^{(n+1)}|$, i.e.,

$$S_1 = \Sigma_1 \big| |K_{(0),1}^{(n+1)}|.$$

Select $q_{1,1}$ to be the first element of $\mathbb{P} \backslash P_G$. For each $\sigma \in S_1^{[n+1]}$, let Θ_σ be a Moore space of type $(\mathbb{Z}/q_{1,1}, n)$ and let $e_\sigma : \partial \sigma \to S_{\Theta_\sigma}^n \subset \Theta_\sigma$ be an embedding. Let $(M_{1,1}, \phi_{1,1})$ be an extension (Definition 5.2) of $S_1^{(n)}$ associated with $\{e_\sigma \,|\, \sigma \in S_1^{[n+1]}\}$. Define

$$M_1 = M_{1,1} \text{ and } \phi_1 = \phi_{1,1} : M_1 \to |S_1|.$$

Select a countable collection $\mathcal{H}_1 = \bigcup \{\mathcal{H}(M_1, H) \,|\, H \in \sigma_0(G)\} = \{h_1^j : A_1^j \to K(H_1^j, n) \,|\, j \in \mathbb{N}\}$ of maps using Lemma 3.12.

Because of (1), it must be true that $u_1 = 1$; consider the map $h_{u_1}^{v_1} = h_1^{v_1} : A_{u_1}^{v_1} = A_1^{v_1} \to K(H_{u_1}^{v_1}, n)$ from the closed subspace $A_{u_1}^{v_1}$ of M_1. Put $C_{1,0} = S_1$. There exists a triangulation $C_{1,1}$ of M_1 such that for each $\sigma \in C_{1,0}^{[n+1]}$, $\phi_{1,1}^{-1}(\sigma)$ is triangulated by $C_{1,1}$ in a standard manner (see the text preceding Notation 5.1). We arrange this so that for some subcomplex N_1 of $C_{1,1}$, $|C_{1,1}^{(n)}| \cup A_{u_1}^{v_1} \subset |N_1|$, and there is a map $\lambda_1 : |N_1| \to K(H_{u_1}^{v_1}, n)$ which is an extension of $h_{u_1}^{v_1}$.

Notice that we have created (Definition 6.1) a 1-layer $\{q_{1,1}\}$-extension $M(C_{1,0}) = \big((M_{1,1}, \phi_{1,1})\big)$, of $S_1^{(n)}$, whose triangulation is $C_{1,1}$. Apply Definition 3.8 with $\Sigma = \Sigma_1$ to get a 1-shift; let $\mathbf{K}_1 = (K_{(1)i}, p_{(1)i}^{i+1})$ denote the resulting inverse sequence.

Now we are ready to make our inductive statement. Suppose $w \in \mathbb{N}$ and we have chosen $1 = r(1) < \cdots < r(w)$ in $\mathbb{N}$ and for each $1 \leq k \leq w$, successive $r(k)$-shifts from Σ_k as in Lemma 3.9. From this we get a finite sequence $\mathbf{K}_0, \ldots, \mathbf{K}_w$ of inverse sequences with the notation (see the text before Definition 3.10) $\mathbf{K}_k = (K_{(k)i}, p_{(k)i}^{i+1})$. We always define

$$S_k = \Sigma_k \big| |K_{(0),r(k)}^{(n+1)}|.$$

Let us require that for each k we have determined a finite CW-complex M_k with $\dim M_k \leq n + 1$, and a map $\phi_k : M_k \to |S_k|$ so that:

(I1) $\operatorname{mesh} \Sigma_k < \lambda^u(\mathcal{M}_{r(k)})/4$;

(I2) if $1 < k$, there is a map $g_{k-1}^k : M_k \to M_{k-1}$ so that

$$d_{r(k-1)}\big(\phi_{k-1} \circ g_{k-1}^k, p_{(k-1)r(k-1)}^{r(k)} \circ \phi_k\big) < \lambda^u(\mathcal{M}_{r(k)})/4;$$

(I3) we have chosen a collection $\mathcal{H}_k = \bigcup \{\mathcal{H}(M_k, H) \,|\, H \in \sigma_0(G)\} = \{h_k^j : A_k^j \to K(H_k^j, n) \,|\, j \in \mathbb{N}\}$ of maps as in Lemma 3.12; and

(I4) when $k < w$ we have selected a map $\Lambda_{k+1} : M_{k+1} \to K(H_{u_k}^{v_k}, n)$ such that $\Lambda_{k+1} | (g_{u_k}^{k+1})^{-1}(A_{u_k}^{v_k}) = h_{u_k}^{v_k} \circ g_{u_k}^{k+1} | (g_{u_k}^{k+1})^{-1}(A_{u_k}^{v_k}).$

Putting $C_{k,0} = S_k$, we require further that for some $\{q_{k,1}, \ldots, q_{k,k}\} \subset \mathbb{N}$, there is a k-layer $\{q_{k,1}, \ldots, q_{k,k}\}$-extension $M(C_{k,0}) = \big((M_{k,1}, \phi_{k,1}), \ldots, (M_{k,k}, \phi_{k,k})\big)$ of $C_{k,0}^{(n)}$ with triangulations $\{C_{k,1}, \ldots, C_{k,k}\}$ such that

- (I5) for some subcomplex N_k of $C_{k,k}$, $|C_{k,k}^{(n)}| \cup (g_{u_k}^k)^{-1}(A_{u_k}^{v_k}) \subset |N_k|$, and there is a map $\lambda_k : |N_k| \to K(H_{u_k}^{v_k}, n)$ which is an extension of $h_{u_k}^{v_k} \circ g_{u_k}^k|(g_{u_k}^k)^{-1}(A_{u_k}^{v_k})$,
- (I6) each $q_{k,i}$ is a composite of elements of $\mathbb{P} \backslash P_G$,
- (I7) $M_k = M_{k,k}$,
- (I8) $\phi_k = \phi_{k,1} \circ \cdots \circ \phi_{k,k} : M_k \to |S_k|$, and
- (I9) if $1 < k$ and $H_{u_{k-1}}^{v_{k-1}} \in \{\mathbb{Z}/p, \mathbb{Z}/p^\infty\}$, then for each $\sigma \in \bigcup_{j=1}^{k-1} C_{k,j-1}^{[n+1]}$, $g_{k-1}^k|\partial\sigma : \partial\sigma \to M_{k-1}$ is a map whose degree is a multiple of p^{k-1}.

The reader may check that we have satisfied the inductive requirements in case $w = 1$. We define the inverse sequence

$$\mathbf{Y}_w = (|S_w| \xleftarrow{p_{(w)r(w)}^{r(w)+1}} |K_{(w)r(w)+1}^*| \xleftarrow{p_{(w)r(w)+1}^{r(w)+2}} |K_{(w)r(w)+2}^*| \leftarrow \cdots),$$

where $K_{(w)r(w)+1}^*$ is the $(n+1)$-skeleton of $(p_{(w)r(w)}^{r(w)+1})^{-1}(S_w)$, $K_{(w)r(w)+2}^*$ is the $(n+1)$-skeleton of $(p_{(w)r(w)+1}^{r(w)+2})^{-1}(K_{(w)r(w)+1}^*)$, and so forth. The bonding maps should of course be the appropriate restrictions of the ones indicated. Lemma 3.9 and Definition 3.6 show that the bonding maps of $\mathbf{Y}_w$ are simplicial surjections. Let $Y = \lim \mathbf{Y}_w$. Since Y embeds in $X = \lim \mathbf{K}_w$, then $\dim_G Y \leq n$. Hence $\mathbf{Y}_w$ meets the conditions (i) and (ii) specified in section 7.

The conditions following (ii) in section 7 and immediately preceding Lemma 7.1 are currently met by the data from the induction as follows:

- (2) $C_0 = C_{w,0}$,
- (3) $M(C_{w,0}) = \big((M_{w,1}, \phi_{w,1}, \ldots, (M_{w,w}, \phi_{w,w}\})\big)$, a w-layer $\{q_{w,1}, \ldots, q_{w,w}\}$-extension of $C_{w,0}$, and
- (4) $N = N_w$ ((iii) of section 7).

If $H_{u_w}^{v_w} = \mathbb{Z}/p$, then choose $q_{w,0} = p^{w+1}$. If $H_{u_w}^{v_w} = \mathbb{Z}/p^\infty$, then note that for some $w_0 \in \mathbb{N}$, $\lambda_w(|N_w|) \subset K(\mathbb{Z}/p^{w_0}, n) \subset K(\mathbb{Z}/p^\infty, n)$. In that case, choose $q_{w,0} = \max\{p^{w+1}, p^{w_0}\}$. Otherwise, let $q_{w,0}$ be the first element of $\mathbb{P} \backslash P_G$. Now let us apply Lemma 7.1 with $\{q_{w,0}, q_{w,1}, \ldots, q_{w,w}\} \subset \mathbb{N}$ and $G = H_{u_w}^{v_w}$. We put $r(w+1) = s$, where $s > r(w)$ comes from Lemma 7.1. Using (a) of that lemma, there is a subdivision Σ_{w+1} (denoted V_0 in Lemma 7.1) of $K_{(w)r(w+1)}$ so that (I1) is true for $k = w+1$. Then with $C_{w+1,0} = S_{w+1}$, find a $(w+1)$-layer $\{q_{w+1,1}, \ldots, q_{w+1,w+1}\}$-extension $M(C_{w+1,0}) = \big((M_{w+1,1}, \phi_{w+1,1}), \ldots, (M_{w+1,w+1}, \phi_{w+1,w+1})\big)$ of $C_{w+1,0}^{(n)}$ with triangulations $C_{w+1,i}$, $1 \leq i \leq w+1$, along with a map $g : M_{w+1,w+1} \to M_w$ as in Lemma 7.1. Define

$$M_{w+1} = M_{w+1,w+1}$$

and put

$$g_w^{w+1} : M_{w+1} \to M_w$$

equal the map g from Lemma 7.1. Define

$$\phi_{w+1} = \phi_{w+1,1} \circ \cdots \circ \phi_{w+1,w+1} : M_{w+1} \to |S_{w+1}|,$$

precisely as required by (I8).

In our application of Lemma 7.1, the complex L_1 is in fact S_w, which is a subcomplex of Σ_w. Thus (I1) for $k = w$ yields that mesh $S_w < \lambda^u(\mathcal{M}_{r(w)})/4$, so, (g) of Lemma 7.1 shows that the commutativity desired in (I2) is true for $k = w+1$. Select a collection $\mathcal{H}_{w+1}$ so that (I3) is satisfied. We have to find a map Λ_{w+1} to meet the condition in (I4).

In case $H_{u_w}^{v_w} = \mathbb{Z}_{(P_G)}$, consider in (I5), for $k = w$, the subcomplex N_w of $C_{w,w}$ and the map $\lambda_w : |N_w| \to K(H_{u_w}^{v_w}, n)$. Simply apply (f) of Lemma 7.1 to get the needed extension Λ_{w+1} in (I4).

When $H_{u_w}^{v_w} \in \{\mathbb{Z}/p, \mathbb{Z}/p^\infty\}$ we need a different approach. This time we shall apply (d) of Lemma 7.1. For $x \in (g_w^{w+1})^{-1}(|N_w|)$ we of course define $\Lambda_{w+1}(x) = \lambda_w \circ g_w^{w+1}(x)$. It remains, according to (d) of Lemma 7.1, to extend Λ_{w+1} to sets of the form $\phi_{w+1,w+1}^{-1}(\sigma')$, where $\sigma' \in C_{w+1,w}^{[n+1]}$ and $\phi_{w+1,w+1}^{-1}(\sigma') \cap (g_w^{w+1})^{-1}(|N_w|) = \partial\sigma'$. But $\phi_{w+1,w+1}^{-1}(\sigma')$ is a Moore space of type $(\mathbb{Z}/q_{w+1,w+1}, n)$, where $q_{w+1,w+1}$ is divisible by $q_{w,0}$. If $H_{u_w}^{v_w} = \mathbb{Z}/p$, then from the fact that $q_{w,0}$ is divisible by p, we see that Λ_{w+1} extends to a map of $\phi_{w+1,w+1}^{-1}(\sigma')$ to $K(H_{u_w}^{v_w}, n)$. If $H_{u_w}^{v_w} = \mathbb{Z}/p^\infty$, then since $\lambda_w(|N_w|) \subset K(\mathbb{Z}/p^{w_0}, n)$ and $q_{w,0}$ is divisible by p^{w_0}, we get the needed extension in this situation also. So (I4) is satisfied.

We have already chosen the extension $M(C_{w+1,0})$ and the triangulations $\{C_{w+1,1}, \ldots, C_{w+1,w+1}\}$, but if we take just a bit more care in choosing $C_{w+1,w+1}$, we can be certain of the existence of a subcomplex N_{w+1} and a map λ_{w+1} as required by (I5). Both (I6) and (I7) can easily be verified from our construction. What remains is (I9). This, however, follows from (c) of Lemma 7.1 and the fact that under these choices of the group $H_{u_w}^{v_w}$, $q_{w,0}$ is divisible by p^{w+1}.

Our inductive construction is complete.

We have selected a sequence $(\mathbf{K}_i)$ satisfying the conditions of Definition 3.10, so let $\mathbf{K}_\infty$ denote its limit. Using Lemma 3.11, one sees that $X = \lim \mathbf{K}_\infty$. To prepare for the use of Lemma 3.14, for each $i \in \mathbb{N}$ set

(5) $\tilde{K}_i = |K_{(0)r(i)}|$, $\tilde{\mathcal{M}}_i = \mathcal{M}_{r(i)}$, $\tilde{p}_i^{i+1} = p_{(i)r(i)}^{r(i+1)}$, and $\mathbf{X} = (\tilde{K}_i, \tilde{p}_i^{i+1})$.

We have already constructed (M_i, g_i^{i+1}) and $\phi_i : M_i \to \tilde{K}_i$.

There is no loss of generality in taking $X = \lim \mathbf{X}$. For each $i \in \mathbb{N}$ and $x \in \tilde{K}_i$, by the definition of λ^u, there exists $M_{x,i} \in \tilde{\mathcal{M}}_i = \overline{\mathrm{st}}K_{(0)r(i)}$ as required in Lemma 3.14. For a given $M = M_{x,i} \in \tilde{\mathcal{M}}_i$, conditions (f) and (b) of Lemma 6.2 imply that each vertex in M lies in $\phi_i^{-1}(M)$; so (a) of Lemma 3.14 is satisfied. We get (b) from the first part of (I2). By Definition 3.6, $\mathbf{K}_0$ satisfies (c) of Lemma 3.4, so an application of Lemma 3.5 gives us the condition (c) of Lemma 3.14. Thus there is a surjective map $\pi : Z = \lim \mathbf{Z} \to X = \lim \mathbf{X}$ whose fibers $\lim \mathbf{Z}_x$ are as indicated in Lemma 3.14.

We claim that the metrizable compactum Z and the surjective map $\pi : Z \to X$ will satisfy the list (a)–(c) of Theorem 1.1. To begin, since $\dim M_i \leq n+1$ for each i, it is a standard fact that the limit Z of the sequence of M_i's has $\dim \leq n+1$. So (b) is true.

For (c) we only have to show that $\dim_G Z \leq n$ because of (b) of the theorem (the part we just proved) and Lemma 4.11. We shall prove this by using Lemma 4.9. Let $H \in \sigma_0(G)$. The idea is to use Lemma 3.13 with the countable abelian group H and the inverse sequence $\mathbf{Z}$ defining the space Z. So let $i \in \mathbb{N}$ and consider an element of $\mathcal{H}(M_i, H)$, say $h_i^j : A_i^j \to K(H_i^j, n)$ as in (I3). There exists $k_0 \geq i$

so that $\mu(k_0) = (i, j)$; thus $i = u_{k_0}$ and $j = v_{k_0}$. We just apply (I4) to obtain $k = k_0 + 1$ leading to the extension requested in Lemma 3.13.

We have to prove (a) of Theorem 1.1. Fix $x = (x_i) \in X$; we know from Lemma 3.14 (look for $\mathbf{Z}_x$) that $\pi^{-1}(x)$ can be written as the limit of an inverse sequence of the form $(\phi_i^{-1}(M_{x_i,i}), g_i^{i+1})$, where $M_{x_i,i} = \overline{\mathrm{st}}(w_i, K_{(0)r(i)}) \in \tilde{\mathcal{M}}_i$ for some vertex w_i of $K_{(0)r(i)}$. But since the image of ϕ_i is $|S_i| = |K_{(0)r(i)}^{(n+1)}|$, then we may replace $M_{x_i,i}$ by $F_i = \overline{\mathrm{st}}(w_i, K_{(0)r(i)}^{(n+1)})$ in the preceding formulation. This means that $\pi^{-1}(x) = \lim(\phi_i^{-1}(F_i), g_i^{i+1})$.

Thinking of F_i as a CW-complex, one sees that $F_i^{(n)}$ is homotopy equivalent to the union of the n-skeleta of the $(n+1)$-simplexes of $K_{(0)r(i)}$ contained in it. For each such $(n+1)$-simplex σ, S_i induces a triangulation S_σ on σ. Observe also that F_i may be treated, up to homotopy equivalence, as a bouquet of $(n+1)$-cells in 1-1 correspondence with the $(n+1)$-simplexes σ just mentioned. From our construction, one can ascertain that $\phi_i^{-1}(F_i)$ is, up to homotopy equivalence, a bouquet, say B_i, of the objects $\phi_i^{-1}(\sigma)$. According to the last part of Lemma 6.3, each of the objects $[\phi_i^{-1}(\sigma)]^{(n)}$ has trivial homotopy groups in the dimensions $0 \le k < n$. It follows then that $\pi_k(B_i^{(n)}) = 0$ for $0 \le k < n$. From this information we may conclude that for each $i \in \mathbb{N}$ and $0 \le k < n$,

(6) the inclusion $B_i^{(k)} \hookrightarrow B_i^{(k+1)}$ is nullhomotopic.

So it is not difficult to show that for $0 < k < n$ and any abelian group H, every map of $\phi_i^{-1}(F_i)$ to $K(H, k)$ is nullhomotopic. Hence one concludes that

(7) $\tilde{\check{H}}^k(\pi^{-1}(x); H) = 0$ for $0 \le k < n$ and all abelian groups H.

For dimension n the strategy is a bit different. We shall show that

(8) if $p \in \mathbb{P}$ and either Z/p or $\mathbb{Z}/p^\infty$ lies in $\sigma_0(G)$, then for all $j \in \mathbb{N}$, $\check{H}^n(\pi^{-1}(x); \mathbb{Z}/p^j) = 0$.

Then surely $\check{H}^n(\pi^{-1}(x); \mathbb{Z}/p) = 0$, but also

(9) $\check{H}^n(\pi^{-1}(x); \mathbb{Z}/p^\infty) = 0$

by the following reasoning. Let us recall that for any compact metrizable space P and abelian group H, $\check{H}^n(P; H) = 0$ if and only if every map of P to $K(H, n)$ is nullhomotopic. Since $\pi^{-1}(x)$ is compact, every map of it to $K(\mathbb{Z}/p^\infty, n)$ lands in some $K(\mathbb{Z}/p^j, n)$. From (8), we see that such a map must be nullhomotopic as a map into $K(\mathbb{Z}/p^j, n) \subset K(\mathbb{Z}/p^\infty, n)$.

Let us assume that $p \in \mathbb{P}$ and either $\mathbb{Z}/p$ or $\mathbb{Z}/p^\infty \in \sigma_0(G)$. Fix $j \in \mathbb{N}$ and let H be either $\mathbb{Z}/p^j$ or a torsion free group. We are going to show that

(10) for all $k \in \mathbb{N}$, there exists $i \ge k$ such that for any map $\varphi : \phi_i^{-1}(F_i) \to K(H, n)$, the composition $\varphi \circ g_i^{i+1} : \phi_{i+1}^{-1}(F_{i+1}) \to K(H, n)$ is nullhomotopic.

This will show that the induced homomorphism

$$H^n(g_i^{i+1}) : H^n(\phi_i^{-1}(F_i); H) \to H^n(\phi_{i+1}^{-1}(F_{i+1}); H)$$

is trivial. That would surely prove (8), but at the same time it would yield

(11) $\check{H}^n(\pi^{-1}(x); H) = 0$ for any torsion free abelian group H.

Thus we are going to establish (10).

We claim that

(12) for all $k \in \mathbb{N}$, there exists $i \geq k$ such that for any map $h : S^n \to \phi_{i+1}^{-1}(F_{i+1})$,
the composition $\varphi \circ g_i^{i+1} \circ h : S^n \to K(H, n)$ is nullhomotopic.

Let us see why (12) would imply (10) and then see how to prove (12).

Because of (6), there exists a homotopy $G^* : B_{i+1}^{(n-1)} \times I \to B_{i+1}^{(n)}$, where G_0^* is the inclusion and G_1^* is a constant map, say to p_0. Let τ be an n-cell of B_{i+1}. We determine a map $d_\tau : \partial(\tau \times I) \to B_{i+1}^{(n)}$ by setting $d_\tau(t, s) = t$ if $s = 0$, $G^*(t, s)$ if $t \in \partial\tau$, and p_0 if $s = 1$. Then the map $\varphi \circ g_i^{i+1} \circ d_\tau : \partial(\tau \times I) \to K(H, n)$ is nullhomotopic by (12). So the map $\varphi \circ g_i^{i+1} \circ d_\tau$ extends to a map $D_\tau : \tau \times I \to K(\mathbb{Z}/p^j, n)$. A map $D : B_{i+1}^{(n)} \times I \to K(H, n)$ is defined by setting $D = \bigcup \{D_\tau \,|\, \tau \text{ an } n\text{-cell of } B_{i+1}\}$. One notes that D is a nullhomotopy of the map $\varphi \circ g_i^{i+1} | B_{i+1}^{(n)}$. Since $\pi_{n+1}(K(H, n)) = 0$, this nullhomotopy extends to one of $B_{i+1}^{(n+1)} \times I$ to $K(H, n)$. This proves (10) modulo (12).

Now for (12). In case H is a torsion free group, e.g., $H = \mathbb{Z}_{(P_G)}$ in our case, then choose $i = k$. It follows from (I7) and Lemma 6.3 that $\pi_n(\phi_{i+1}^{-1}(F_{i+1}))$ is a torsion group. So we simply have to note that H is a torsion free group.

Suppose that $H = \mathbb{Z}/p^j$; then we may assume without loss of generality that $k \geq j$. There exists l such that $H_k^l \in \{\mathbb{Z}/p, \mathbb{Z}/p^\infty\}$. Hence there is $i \geq k \geq j$ such that $\mu(i) = (k, l)$, i.e., $u_i = k$ and $v_i = l$. From (I9), one sees that for each $\sigma \in \bigcup_{j_0=1}^{i} C_{i+1, j_0-1}^{[n+1]}$, g_i^{i+1} carries $\partial\sigma$ to M_i by a map whose degree is a multiple of p^j. So it must be true that $\varphi \circ g_i^{i+1}$ carries $\partial\sigma$ to $K(\mathbb{Z}/p^j, n)$ by a map whose degree is a multiple of p^j. This and an application of Lemma 6.3 show that for any map h as in (12), the composition $\varphi \circ g_i^{i+1} \circ h$ has degree which is a multiple of p^j. Such a map has to be nullhomotopic. Our proof of (12) is complete.

Now we shall establish that each fiber B of the map π is G-acyclic. We already have proved that $\dim_G Z \leq n$, so $\dim_G B \leq n$, that is, (a) of Lemma 4.10 is certainly true. One obtains (b), (c), and (d) from (7), (8), (9), and (11).

Since $\pi^{-1}(x) \subset Z$ and $\dim_G Z \leq n$, then $\dim_G(\pi^{-1}(x)) \leq n$. Hence an application of Theorem 1 of [Ku] shows that $\check{H}^{n+k}(\pi^{-1}(x); G) = 0$ for all $k \geq 1$. This completes our proof of (a) of Theorem 1.1. In case $\pi_{n+1}(K) = 0$, then (aa) is assured by an application of Lemma 4.12 since each fiber of π is G-acyclic and has $\dim \leq n + 1$. $\square$

References

[Dr1] A. Dranishnikov, *On P. S. Aleksandrov's problem*, Mat. Sb. **135 (4)** (1988), 551–557. MR0942139 (90e:55004)

[Dr2] A. Dranishnikov, *On homological dimension modulo p*, Math. USSR Sbornik **60, no. 2** (1988), 413–425. MR0889602 (88h:55001)

[Dr3] A. Dranishnikov, *Homological dimension theory*, Russian Math. Surveys **43(4)** (1988), 11–63. MR0969565 (90e:55003)

[Dr4] A. Dranishnikov, *K-theory of Eilenberg-Mac Lane spaces and cell-like mapping problem*, Trans. Amer. Math. Soc. **335:1** (1993), 91–103. MR1153012 (93e:55003)

[Dr5] A. Dranishnikov, *Rational homology manifolds and rational resolutions*, Topology and its Appls. **94** (1999), 75–86. MR1695349 (2000j:55001)

[Dr6] A. Dranishnikov, *Cohomological dimension theory of compact metric spaces*, preprint.

[DD] A. Dranishnikov and J. Dydak, *Extension dimension and extension types*, Tr. Mat. Inst. Steklova **212** (1996), 61–94. MR1635023 (99h:54049)

[Dy] J. Dydak, *Cohomological dimension theory*, Handbook of Geometric Topology, Elsevier, Amsterdam, 2002, pp. 423–470. MR1886675 (2003a:55002)

[DW] J. Dydak and J. Walsh, *Complexes that arise in cohomological dimension theory: a unified approach*, J. London Math. Soc. **(2) (48), no. 2** (1993), 329–347. MR1231719 (94d:55003)

[Ed] R. D. Edwards, *A theorem and question related to cohomological dimension and cell-like maps*, Notices of the Amer. Math. Soc. **25** (1978), A-259.

[Hu] S. Hu, *Homotopy Theory*, Academic Press, New York, 1959. MR0106454 (21:5186)

[JR] R. Jimenez and L. Rubin, *An addition theorem for n-fundamental dimension in metric compacta*, Topology and its Appls. **62** (1995), 281–297. MR1326827 (96a:54026)

[KY1] A. Koyama and K. Yokoi, *A unified approach of characterizations and resolutions for cohomological dimension modulo p*, Tsukuba J. Math. **18(2)** (1994), 247–282. MR1305817 (95j:55002)

[KY2] A. Koyama and K. Yokoi, *Cohomological dimension and acyclic resolutions*, Topology and its Appls. **120** (2002), 175–204. MR1895491 (2003b:55001)

[Ku] W. I. Kuz'minov, *Homological Dimension Theory*, Russian Math. Surveys **23** (1968), 1–45. MR0240813 (39:2158)

[Le] M. Levin, *Acyclic resolutions for arbitrary groups*, Isr. J. Math. **135** (2003), 193–204. MR1997043 (2004f:55001)

[Ma] S. Mardešić, *Extension dimension of inverse limits*, Glasnik Mat. **35(55)** (2000), 339–354. MR1812562 (2001i:54038)

[MR] S. Mardešić and L. Rubin, *Cell-like mappings and nonmetrizable compacta of finite cohomological dimension*, Trans. Amer. Math. Soc. **313** (1989), 53–79. MR0962284 (90a:54095)

[MS1] S. Mardešić and J. Segal, *Shape Theory*, North-Holland, Amsterdam, 1982. MR0676973 (84b:55020)

[MS2] S. Mardešić and J. Segal, *Stability of almost commutative inverse systems of compacta*, Topology and its Appls. **31** (1989), 285–299. MR0997496 (90e:54018)

[Ru1] L. Rubin, *Cell-like maps, dimension and cohomological dimension: a survey*, Banach Center Publications **18** (1986), 371–376. MR0925877 (89a:55001)

[Ru2] L. Rubin, *Cohomological dimension and approximate limits*, Proc. Amer. Math. Soc. **125** (1997), 3125–3128. MR1423333 (98g:55001)

[Ru3] L. Rubin, *A stronger limit theorem in extension theory*, Glasnik Mat. **36(56)** (2001), 95–103. MR1842828 (2002c:54031)

[RS] L. Rubin and P. Schapiro, *Cell-like maps onto non-compact spaces of finite cohomological dimension*, Topology and its Appls. **27** (1987), 221–244. MR0918533 (89b:55002)

[Sh] E. Shchepin, *Arithmetic of dimension theory*, Russian Math. Surv. **53** (1998), 975–1069. MR1691185 (2002a:55002)

[Sp] E. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966. MR0210112 (35:1007)

[Wa] J. Walsh, *Dimension, cohomological dimension, and cell-like mappings*, Shape Theory and Geometric Topology, Lecture Notes in Mathematics, volume 870, Springer Verlag, Berlin, 1981, pp. 105–118. MR0643526 (83a:57021)

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